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Vektoranalysis (für PhysikerInnen) — SS 2014

Wiederholungsblatt (ohne Wertung) — 13. Mai

Aufgabe 1
Skizzieren Sie die durch die folgenden Mengen gegebenen Kurven und geben Sie jeweils eine (möglichst einfache) Parametrisierung an. Bestimmen Sie die Länge von C_1 und C_2.

(i) $C_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 1 + 2y = 1 + z, 0 \leq x \leq 2\}$;

(ii) $C_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - \lambda)^2 = r^2\}$ für feste $\lambda, r \in \mathbb{R}$;

(iii) $C_3 = \{(x, y) \in \mathbb{R}^2 : xy = \lambda, 1 \leq y \leq 3\}$ für festes $\lambda \in \mathbb{R}$.

Aufgabe 2
Es sei $f : \mathbb{R}^2 \to \mathbb{R}$ stetig. Kehren Sie bei den folgenden Integralen die Integrationsreihenfolge um.

(i) $\int_0^1 \int_0^x f(x, y)dydx$;

(ii) $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x, y)dydx$;

(iii) $\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x, y)dxdy$.

Aufgabe 3
Skizzieren Sie jeweils die Menge A und berechnen Sie das Integral $\int_A f(x, y, z)d(x, y, z)$.

(i) $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto xyz$ mit $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, x, y, z \geq 0\}$ für festes $R > 0$.

(ii) $f : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto \cos(z/y)$ mit $A = \{(x, y, z) \in \mathbb{R}^3 : \pi/6 \leq y \leq \pi/2, y \leq x \leq \pi/2, 0 \leq z \leq xy\}$.

Aufgabe 4
Es seien $0 < R_1 < R_2$. Skizzieren Sie die Menge

$$T := \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - R_2\right)^2 + z^2 \leq R_1^2\right\}$$

und berechnen Sie das Volumen von T.

Aufgabe 5
In einem Fluss mit konstanter Strömungsgeschwindigkeit $v(x, y, z) = (2, 0, 0)$ sei ein Netz aufgehängt, dessen Form durch die Abbildung

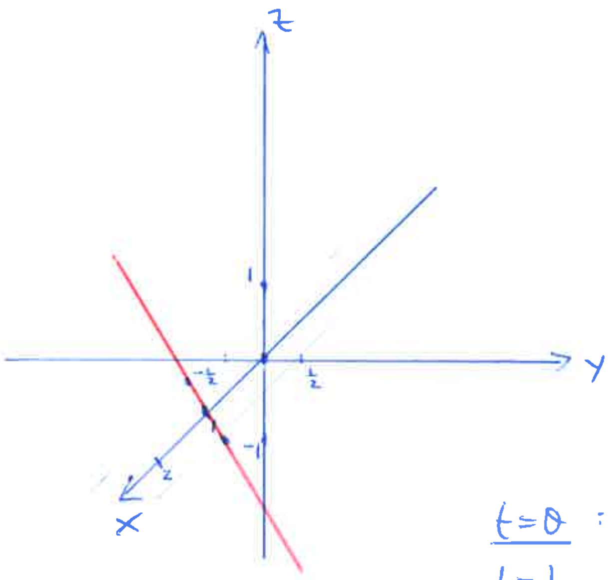
$$p : [0, 3] \times [0, 2\pi] \to \mathbb{R}^3, \quad (u, v) \mapsto \left(u - \tanh u, \frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}\right)$$

gegeben ist. Berechnen Sie den Gesamtfluss des Wassers durch das Netz pro Zeiteinheit.

A Winding

Aufgabe 1

(i) $C_1 = \{(x,y,z) \in \mathbb{R}^3 : x = 1 + 2y,\ x = z + 1,\ 0 \leq x \leq 2\}$



Parametrization:

$t \in [0,2]$

$x = t$

$y = \frac{1}{2}(t-1)$

$z = t - 1$

$t=0$: $x = 0,\ y = -\frac{1}{2},\ z = -1$

$t=1$: $x = 1,\ y = 0,\ z = 0$

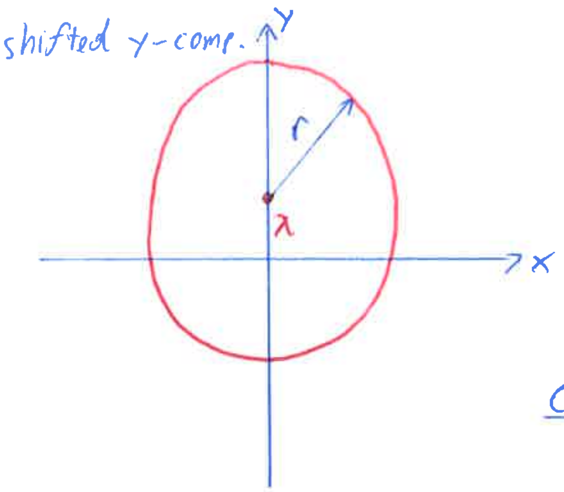
$t=2$: $x = 2,\ y = \frac{1}{2},\ z = 1$

calculate length:

$$\ell = \int_0^2 dt\ \left[(x'(t))^2 + (y'(t))^2 + (z'(t))^2 \right]^{\frac{1}{2}}$$

$$= \int_0^2 dt\ \left[1 + \frac{1}{4} + 1 \right]^{\frac{1}{2}} = \frac{3}{2} \int_0^2 dt = \underline{\underline{3}}$$

(ii) $C_2 = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-\lambda)^2 = r^2;\ \lambda, r \in \mathbb{R},\ \text{const.}\}$



Parametrization

r, λ fixed $\Rightarrow$ φ is Parameter

$\varphi \in [0, 2\pi)$

$x = r\cos\varphi$

$y = r\sin\varphi + \lambda$

Check: $x^2 + (y-\lambda)^2 = r^2\cos^2\varphi + (r\sin\varphi + \lambda - \lambda)^2$

$= r^2\cos^2\varphi + r^2\sin^2\varphi = \underline{\underline{r^2}}$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\, d\varphi$$

here: $\quad r = \text{const} \Rightarrow \quad ds = \sqrt{r^2}\, d\varphi = r\, d\varphi$

$$\Rightarrow \quad l = \int_0^{2\pi} d\varphi\, r = \underline{\underline{2r\pi}}$$

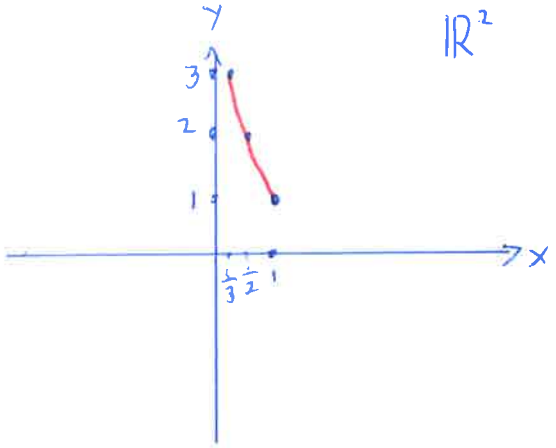
(iii) $\quad C_3 = \{ (x,y) \in \mathbb{R}^2 ;\ xy = 1,\ 1 \leq y \leq 3 \},\quad \lambda \text{ const.},\ \lambda \in \mathbb{R}$

Parametrization

$t \in [1, 3]$ $\qquad$ $t=1$: $x = 1,\ y = 1$

$x = \frac{1}{t}$ $\qquad$ $t=2$: $x = \frac{1}{2},\ y = 2$

$y = t$ $\qquad$ $t=3$: $x = \frac{1}{3},\ y = 3$



arclength (for sake of completeness):

$$l = \int_1^3 dt\, \left[(x'(t))^2 + (y'(t))^2 \right]^{\frac{1}{2}}$$

$$= \int_1^3 dt\, \left[\frac{1}{t^4} + 1 \right]^{\frac{1}{2}} = \left| \text{use mathematica} \right|$$

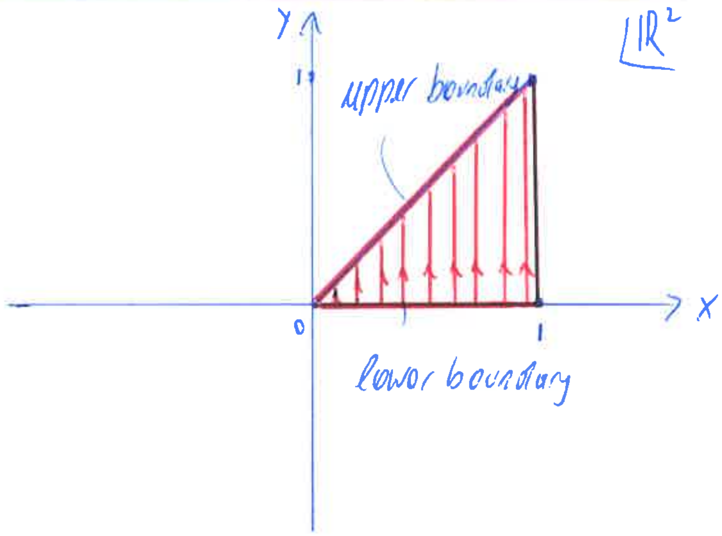
$$= \frac{1}{3} \frac{\sqrt{2\pi}\, \Gamma\left[\frac{7}{4}\right]}{\Gamma\left[\frac{5}{4}\right]} - {}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{4}, \tfrac{3}{4}, -81\right) \approx \underline{\underline{2.1466}}$$

Aufgabe 2

$f: \mathbb{R}^2 \to \mathbb{R}$ continuous.

(i) $I_1 = \int_0^1 dx \int_0^x dy \; f(x,y)$

consider region of integration:

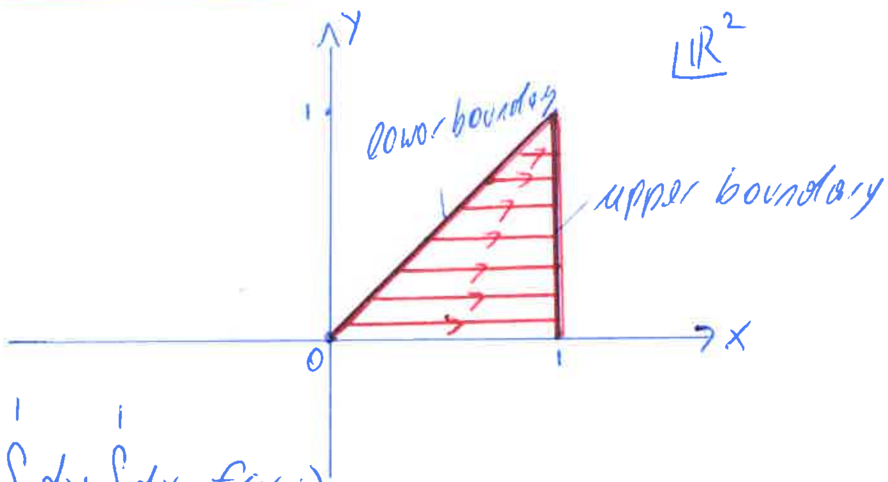


exchange order of integration:

$I_1 = \int_0^1 dy \int_a^b dx \; f(x,y)$

choose a and b such that the region of integration is preserved.

Obvious choice: $a = y$, $b = 1$

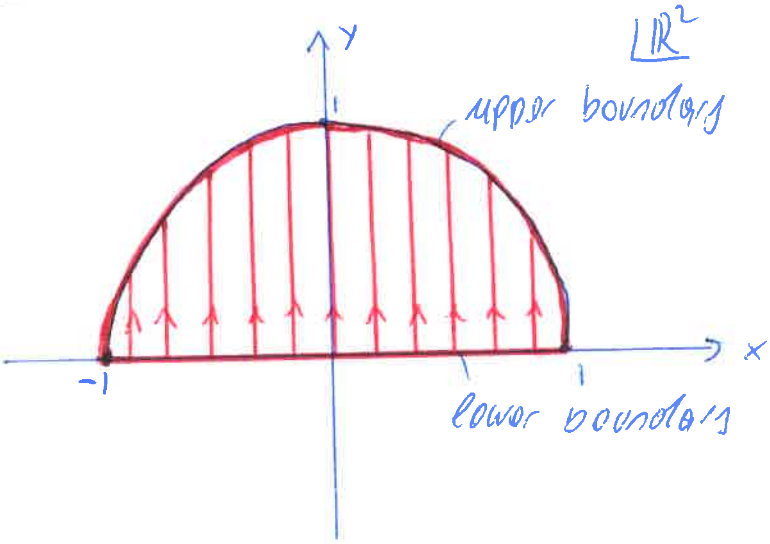


$$I_1 = \int_0^1 dy \int_y^1 dx \; f(x,y)$$

(ii) $f: \mathbb{R}^2 \to \mathbb{R}$ continuous

$$I_2 = \int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} dy\, f(x,y)$$

consider region of integration:

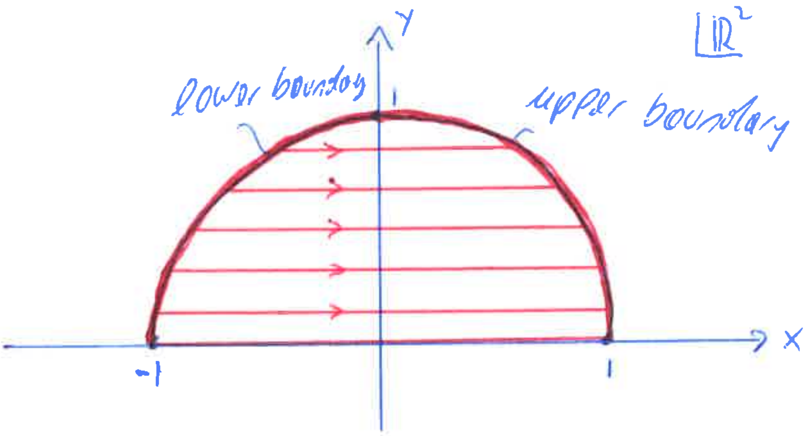


exchange order of integration:

$$I_2 = \int_{0}^{1} dy \int_{a}^{b} dx\, f(x,y)$$

choose a and b such that region of integration is preserved.

$$\Rightarrow \quad a = -\sqrt{1-y^2}\ , \ b = \sqrt{1-y^2}$$



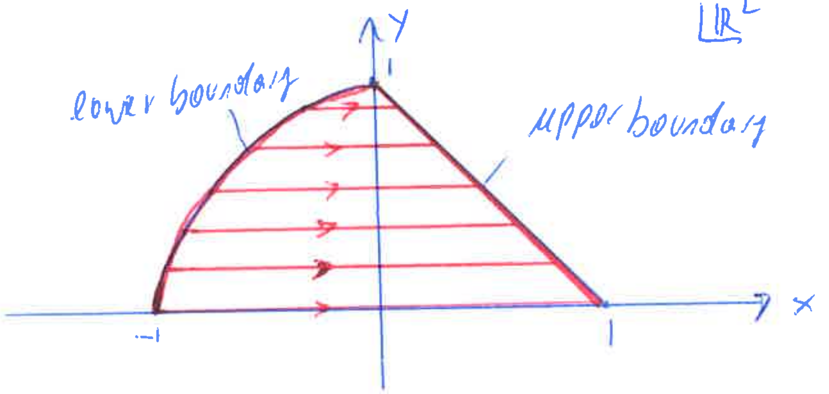
$$I_2 = \int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\, f(x,y)$$

(iii)

$f: \mathbb{R}^2 \to \mathbb{R}$ continuous.

$$I_3 = \int_0^1 dy \int_{-\sqrt{1-y^2}}^{1-y} dx \; f(x,y)$$

consider region of integration:



exchange order of integration: (split region because upper bound changes)

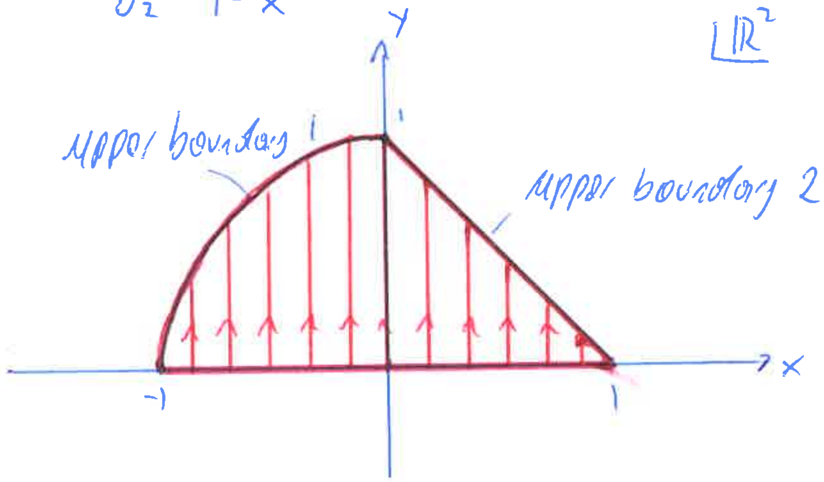
$$I_3 = \int_{-1}^{0} dx \int_{a_1}^{b_1} dy \; f(x,y) + \int_0^1 dx \int_{a_2}^{b_2} dy \; f(x,y)$$

choose boundaries such that region of integration is preserved

$\Rightarrow \quad a_1 = a_2 = 0$

$b_1 = \sqrt{1-x^2}$

$b_2 = 1-x$



$$I_3 = \int_{-1}^{0} dx \int_0^{\sqrt{1-x^2}} dy \; f(x,y) + \int_0^1 dx \int_0^{1-x} dy \; f(x,y)$$

Aufgabe 3

evaluate $\int_A f(x,y,z)\, dx\, dy\, dz$

(1) $f: \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto xyz$ $\qquad A = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq R^2,\ x,y,z \geq 0\}$

$R > 0$, const.

use spherical coordinates:

$x = r \sin\vartheta \cos\varphi$

$y = r \sin\vartheta \sin\varphi$

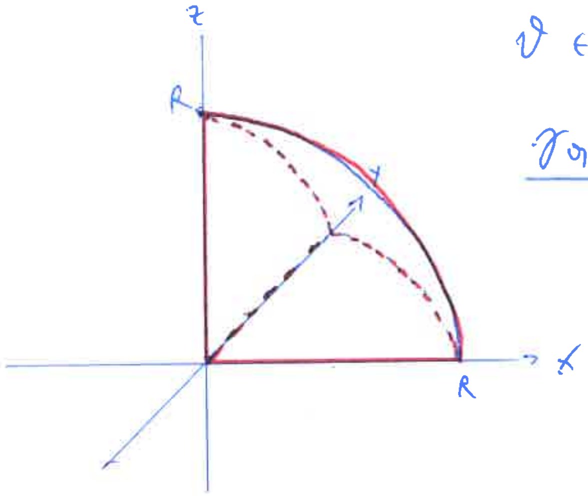
$z = r \cos\vartheta$

here boundaries:

$r \in [0, R]$

$\varphi \in [0, \frac{\pi}{2}]$

$\vartheta \in [0, \frac{\pi}{2}]$



Jacobian:

$dx\,dy\,dz = r^2 \sin\vartheta\, dr\, d\vartheta\, d\varphi$

$x \cdot y \cdot z = r^3 \cos\vartheta \sin^2\vartheta \cos\varphi \sin\varphi$

$$\Rightarrow \int_A f(x,y,z)\, dx\, dy\, dz = \int_0^{\pi/2} d\varphi \int_0^R dr \int_0^{\pi/2} d\vartheta\, r^5 \sin^3\vartheta \cos\vartheta \cos\varphi \sin\varphi$$

$$= \frac{R^6}{6} \int_0^{\pi/2} d\varphi \cos\varphi \sin\varphi \int_0^{\frac{\pi}{2}} d\vartheta \sin^3\vartheta \cos\vartheta$$

$$\int_0^{\frac{\pi}{2}} d\varphi \cos\varphi \sin\varphi = \left| \begin{array}{l} u = \sin\varphi \\ du = \cos\varphi\, d\varphi \\ 0 \to 0 \\ \frac{\pi}{2} \to 1 \end{array} \right| = \int_0^1 \frac{du}{\cancel{\cos\varphi}} \cancel{\cos\varphi}\, u = \frac{u^2}{2}\Big|_0^1 = \underline{\underline{\frac{1}{2}}}$$

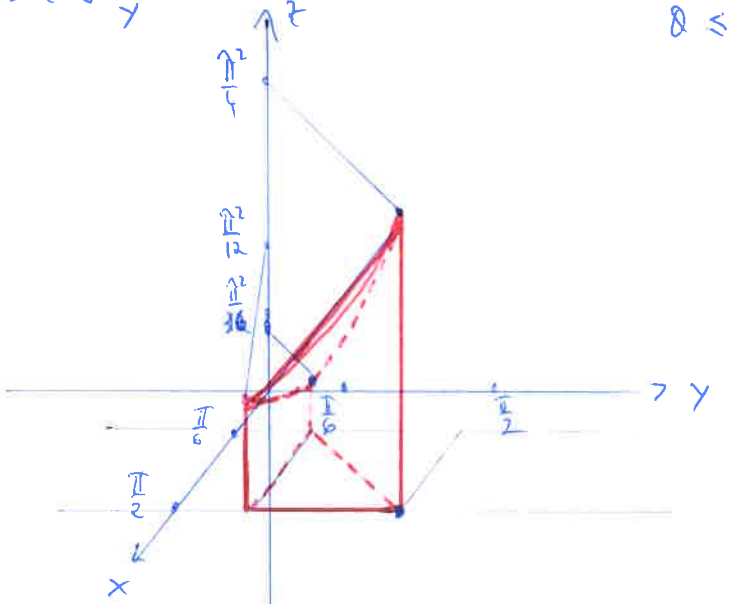
$$\int_0^{\frac{\pi}{2}} d\vartheta \sin^3\vartheta \cos\vartheta = \left| \begin{array}{l} u = \sin\vartheta \\ du = \cos\vartheta\, d\vartheta \\ 0 \to 0 \\ \frac{\pi}{2} \to 1 \end{array} \right| = \int_0^1 \frac{du}{\cancel{\cos\vartheta}} \cancel{\cos\vartheta}\, u^3 = \frac{u^4}{4}\Big|_0^1 = \underline{\underline{\frac{1}{4}}}$$

⑦

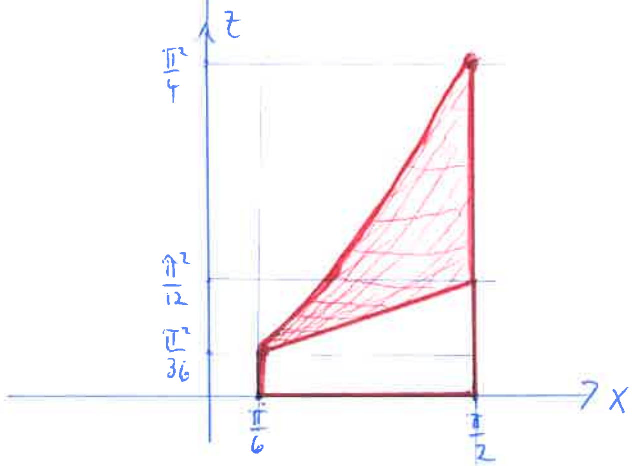
$$\Rightarrow \frac{R^6}{6}\int_0^{\frac{\pi}{2}} d\varphi \cos\varphi \sin\varphi \int_0^{\frac{\pi}{2}} d\vartheta \sin^3\vartheta \cos\vartheta = \frac{R^6}{6}\cdot\frac{1}{2}\cdot\frac{1}{4} = \underline{\underline{\frac{R^6}{48}}}$$

(ii) $f: \mathbb{R}^3 \to \mathbb{R}$
$(x,y,z) \mapsto \cos\frac{z}{y}$

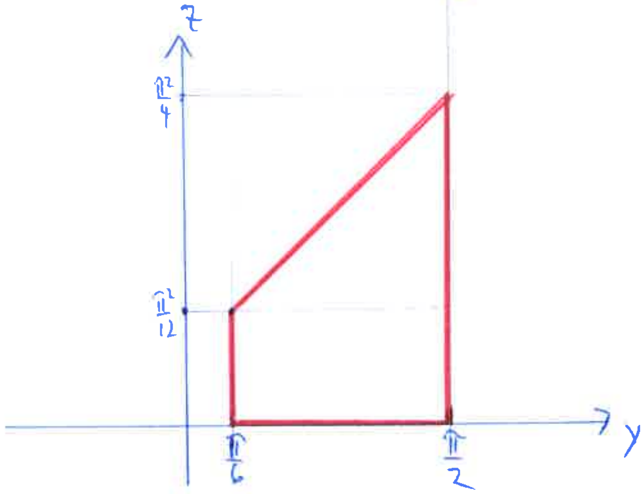
$A = \{(x,y,z)\in\mathbb{R}^3 : \frac{\pi}{6} \leq y \leq \frac{\pi}{2},\ y \leq x \leq \frac{\pi}{2},\ 0 \leq z \leq xy\}$



xz-view:



yz-view



Integration:

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} dy \int_{y}^{\frac{\pi}{2}} dx \int_{0}^{xy} dz \cos\frac{z}{y} = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} dy \int_{y}^{\frac{\pi}{2}} dx \; y \sin\frac{z}{y}\Big|_{0}^{xy}$$

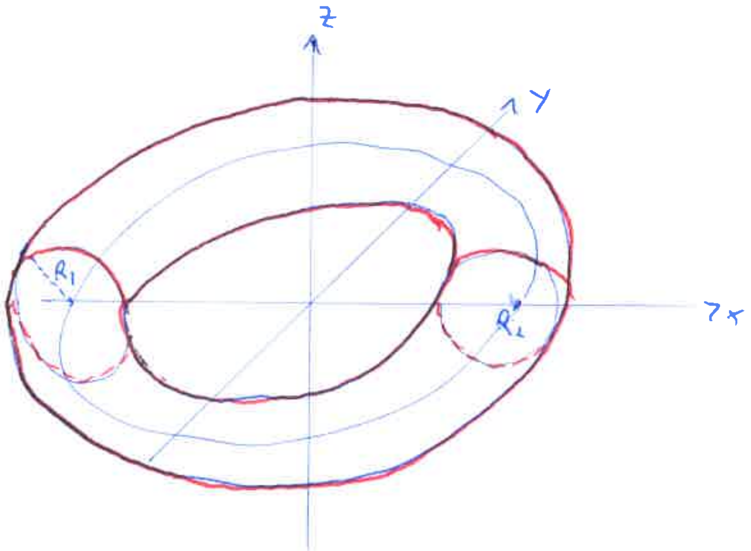
$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} dy \int_{y}^{\frac{\pi}{2}} dx \; y \sin x = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} dy \; y \, [\cos y] = y \sin y \Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \sin y \, dy$$

$$= \frac{\pi}{6}\frac{1}{2} - \frac{\pi}{2} + \underbrace{\cos\frac{\pi}{6}}_{\frac{\sqrt{3}}{2}} = \frac{\pi}{12} - \frac{\pi}{2} + \frac{\sqrt{3}}{2} = \underline{\underline{-\frac{5\pi}{12} + \frac{\sqrt{3}}{2}}} \sim -0.44$$

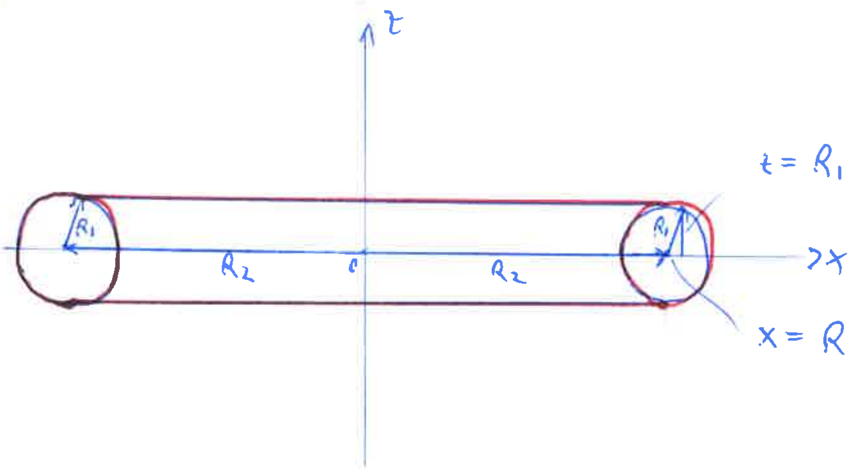
Aufgabe 4

$$T := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R_2)^2 + z^2 \leq R_1^2\} \qquad 0 < R_1 < R_2$$

torus:



section:



$z = R_1 \cdot \sin\vartheta$

$x = R_2 + R_1 \cdot \cos\vartheta$

add polar angle:

$$x = (R_2 + r\cos\vartheta)\cos\varphi$$
$$y = (R_2 + r\cos\vartheta)\sin\varphi$$
$$z = r\sin\vartheta$$

$r \in [0, R_1]$, $\vartheta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$

$$V = \int_0^{R_1} dr \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \; |\det J|$$

$$|\det J| = \left| \begin{vmatrix} \partial_r x & \partial_\vartheta x & \partial_\varphi x \\ \partial_r y & \partial_\vartheta y & \partial_\varphi y \\ \partial_r z & \partial_\vartheta z & \partial_\varphi z \end{vmatrix} \right| = \left| \begin{vmatrix} \cos\vartheta\cos\varphi & -r\sin\vartheta\cos\varphi & -(R_2+r\cos\vartheta)\sin\varphi \\ \cos\vartheta\sin\varphi & -r\sin\vartheta\sin\varphi & (R_2+r\cos\vartheta)\cos\varphi \\ \sin\vartheta & r\cos\vartheta & 0 \end{vmatrix} \right|$$

$$= \Big| \big[-r\sin^2\vartheta(R_2+r\cos\vartheta)\cos^2\varphi - r\cos^2\vartheta(R_2+r\cos\vartheta)\sin^2\varphi$$
$$- r\cos^2\vartheta(R_2+r\cos\vartheta)\cos^2\varphi - r\sin^2\vartheta(R_2+r\cos\vartheta)\sin^2\varphi \big] \Big|$$

$$= \left| -r\sin^2\vartheta(R_2+r\cos\vartheta) - r\cos^2\vartheta(R_2+r\cos\vartheta) \right| = \underline{\underline{r(R_2+r\cos\vartheta)}}$$

$$V = \int_0^{R_1} dr \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \, [r R_2 + r^2\cos\vartheta]$$

$$= R_2 \frac{R_1^2}{2}(2\pi)^2 + \cancel{\frac{R_1^3}{3} 2\pi \int_0^{\pi} d\vartheta \cos\vartheta}$$

$$= \underline{\underline{2\pi^2 R_2 R_1^2}}$$

Aufgabe 5

$v(x,y,z) = (2,0,0)$

Strategy: (w/o Gauß theorem) evaluate $\int_A \vec{v}\cdot\hat{n}\,dA = \int_A \vec{v}\cdot d\vec{a}$

$P: [0,3]\times[0,2\pi] \to \mathbb{R}^3$

$(u,v) \mapsto \left(u-\tanh u,\ \frac{\cos v}{\cosh u},\ \frac{\sin v}{\cosh u}\right)$

$\Rightarrow\ P(u,v) = \left(u-\tanh u,\ \frac{\cos v}{\cosh u},\ \frac{\sin v}{\cosh u}\right),\ u\in[0,3],\ v\in[0,2\pi)$

$$P'(u,v) = \begin{pmatrix} 1-\frac{1}{\cosh^2 u} & 0 \\ -\cos v\,\frac{\sinh u}{\cosh^2 u} & \frac{-\sin v}{\cosh u} \\ -\sin v\,\frac{\sinh u}{\cosh^2 u} & \frac{\cos v}{\cosh u} \end{pmatrix}$$

$$P'_u(u,v)\times P'_v(u,v) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\cosh^2 u-1}{\cosh^2 u} & \frac{-\cos v\sinh u}{\cosh^2 u} & \frac{-\sin v\sinh u}{\cosh^2 u} \\ 0 & \frac{-\sin v}{\cosh u} & \frac{\cos v}{\cosh u} \end{vmatrix}$$

$$= \begin{pmatrix} \frac{-\cos^2 v\sinh u}{\cosh^3 u} - \frac{\sin^2 v\sinh u}{\cosh^3 u} \\ -\frac{\cos v(\cosh^2 u-1)}{\cosh^3 u} \\ -\frac{\sin v(\cosh^2 u-1)}{\cosh^3 u} \end{pmatrix} = \frac{-1}{\cosh^3 u}\begin{pmatrix} \sinh u \\ \cos v(\cosh^2 u-1) \\ \sin v(\cosh^2 u-1) \end{pmatrix}$$

$$= \underline{\underline{\frac{-1}{\cosh^3 u}\begin{pmatrix} \sinh u \\ \cos v\sinh^2 u \\ \sin v\sinh^2 u \end{pmatrix}}}$$

$$\int \vec{v}\cdot d\vec{a} = \int_0^3 du \int_0^{2\pi} dv\, \frac{-2\sinh u}{\cosh^3 u} = -4\pi\int_0^3 du\,\frac{\sinh u}{\cosh^3 u} = \left|\begin{array}{l} t=\cosh u \\ dt=\sinh u\,du \\ 0\to 1 \\ 3\to\cosh 3\end{array}\right|$$

$$= -4\pi\int_0^{\cosh 3}\frac{1}{t^3}\,dt = 2\pi\,\frac{1}{t^2}\Big|_1^{\cosh 3} = \underline{\underline{2\pi\left[\frac{1}{\cosh^2 3}-1\right]}} = \underline{\underline{-2\pi\tanh^2 3}}$$