Uni Graz
TU Graz

S. Kofler, A. Windisch
M. Holzmann, J. Rohleder, M. Zank

Vektoranalysis (für PhysikerInnen)

SS 2014

9. Übungsblatt

3. Juni

Aufgabe 1

Es sei $B \subset \mathbb{R}^3$ der Bereich, der durch den Zylinder $x^2 + y^2 = 4$ und die Ebenen $z = 0$ und $z = 4$ begrenzt ist. Berechnen Sie

$$\iiint_B e^{x^2+y^2}\, dxdydz \qquad \text{und} \qquad \iint_{\partial B} \begin{pmatrix} z \\ x \\ ze^{x^2+y^2} \end{pmatrix} d\vec{A}.$$

Aufgabe 2

Bestimmen Sie den Fluss des Vektorfelds $\vec{F}(x, y, z) = (xz, yz, -z)$ durch die Oberfläche des geraden Kreiskegels

$$K = \left\{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 1 - \sqrt{x^2 + y^2}\right\}.$$

unter Verwendung des Satzes von Gauß. Vergleichen Sie anschließend mit Ihrem Ergebnis der Aufgabe 5 vom 5. Übungsblatt.

Aufgabe 3

Berechnen Sie die Umkehrtransformation der Zylinderkoordinaten

$$\begin{pmatrix} x(r, \varphi, h) \\ y(r, \varphi, h) \\ z(r, \varphi, h) \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ h \end{pmatrix}, \quad r > 0, \varphi \in [0, 2\pi), z \in \mathbb{R}.$$

Tipp: Achten Sie genau auf den jeweiligen Quadranten in der xy-Ebene; unterscheiden Sie verschiedene Fälle!

Aufgabe 4

Es sei $S := \{(x, y, z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2, z \geq 0\}$. Skizzieren Sie S und integrieren Sie die Funktion

$$f : S \to \mathbb{R}, \quad f(x, y, z) := z\sqrt{\frac{1}{4} + x^2 + y^2},$$

über S.

Aufgabe 5

Der Kegel der Höhe h, dessen Basis den Radius r habe, stehe so auf der xy-Ebene, dass seine Symmetrieachse mit der z-Achse zusammenfällt. Berechnen Sie den Fluss des Vektorfelds

$$v(x, y, z) := (z, x, -3y^2), \quad (x, y, z) \in \mathbb{R}^3,$$

(a) durch den Kegelmantel und

(b) durch den Boden des Kegels.

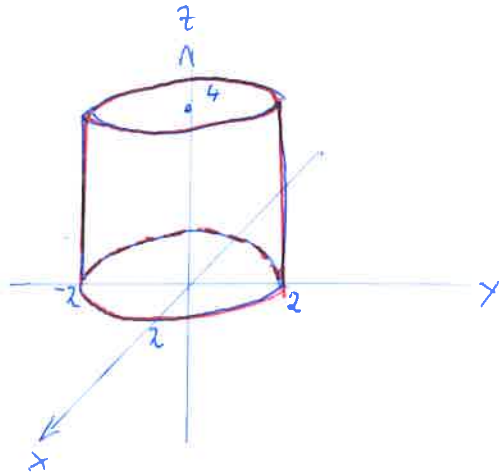
Lösungen zum 9. Übungsblatt aus Vektoranalysis SS14

A. Windisch

Aufgabe 1

$\mathcal{B} \subset \mathbb{R}^3;\quad x^2+y^2=4,\quad z=0,\quad z=4:$

$\mathcal{B} = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 4,\ 0 \leq z \leq 4\}$



- $\displaystyle\iiint_{\mathcal{B}} e^{x^2+y^2}\,dx\,dy\,dz$

use cyl. coords: $\quad \vec{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix},\quad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$$\Rightarrow \int_0^2 dr \int_0^{2\pi} d\varphi \int_0^4 dz\; r\, e^{r^2} = \left| \begin{array}{l} u = r^2 \\ du = 2r\,dr \\ 0 \to 0 \\ 2 \to 4 \end{array} \right| = 2\pi \cdot 4 \int_0^4 \frac{du}{2r}\, r\, e^{u}$$

$$= \underline{\underline{4\pi(e^4-1)}}$$

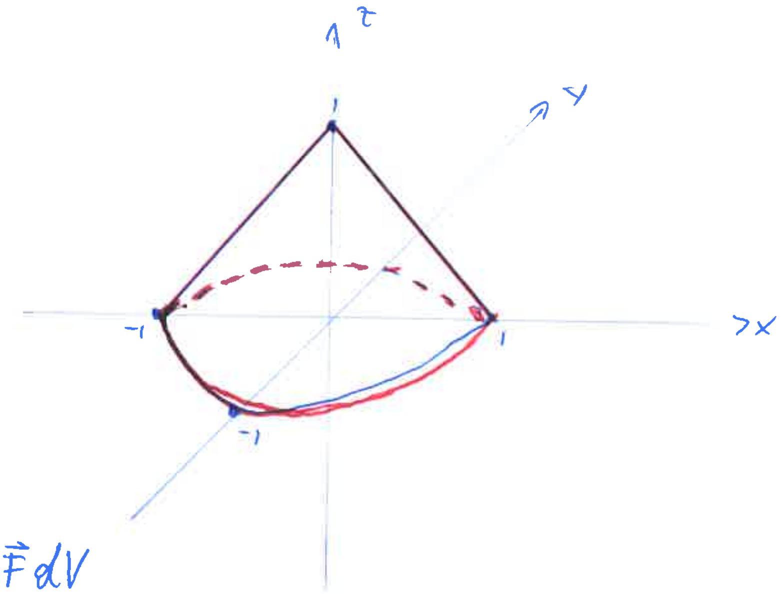
- $$\iint_{\partial\mathcal{B}} \underbrace{\begin{pmatrix} z \\ x \\ z\,e^{x^2+y^2} \end{pmatrix}}_{=\vec{F}} \cdot d\vec{A} = \iiint_{\mathcal{B}} \vec{\nabla}\cdot\vec{F}\,dV$$

$$\underline{\underline{\vec{\nabla}\cdot\vec{F} = e^{x^2+y^2}}} \quad \Rightarrow \quad \iint_{\partial\mathcal{B}} \begin{pmatrix} z \\ x \\ z\,e^{x^2+y^2} \end{pmatrix} \cdot d\vec{A} = \underline{\underline{4\pi(e^4-1)}}$$

Aufgabe 2

$$\vec{F}(x,y,z) = (xz, yz, -z)$$

$$\mathcal{U} = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq z \leq 1 - \sqrt{x^2+y^2}\}$$



$$\oint_{\partial V} \vec{F} \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{F}\, dV$$

cylindrical coords:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\}, \quad \begin{aligned} r &\in [1, 1-z] \\ \varphi &\in [0, 2\pi) \\ z &\in [0,1] \end{aligned}, \quad dV = dx\,dy\,dz = r\,dr\,d\varphi\,dz$$

$$\Rightarrow \quad \underline{\underline{\vec{\nabla} \cdot \vec{F} = 2z - 1}} \quad \Rightarrow$$

$$\int_V \vec{\nabla}\vec{F}\, dV = \int_0^1 dz \int_0^{2\pi} d\varphi \int_1^{1-z} dr\, r(2z-1) =$$

$$= \frac{2\pi}{2} \int_0^1 dz\, (1-z)^2 (2z-1) - \pi \int_0^1 dz\, (2z-1) = \pi \int_0^1 dz \left[-4z^2 + 2z + 2z^3 - z^2\right]$$

$$= \pi \int_0^1 dz \left[2z - 5z^2 + 2z^3\right] = \pi\left[1 - \frac{5}{3} + \frac{1}{2}\right] = \underline{\underline{-\frac{\pi}{6}}}$$

Aufgabe 3

$$\begin{pmatrix} x(r,\varphi,h) \\ y(r,\varphi,h) \\ z(r,\varphi,h) \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ h \end{pmatrix}, \quad r>0,\ \varphi\in[0,2\pi),\ z\in\mathbb{R}$$

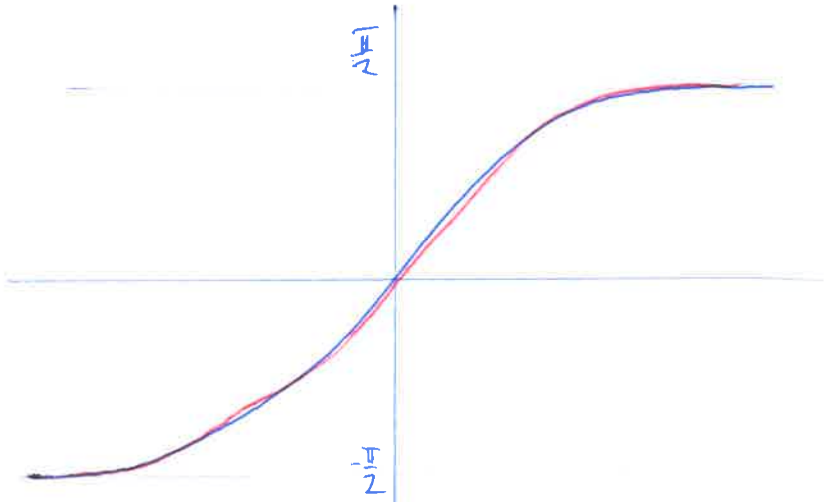
$$\underline{\underline{z = h}}$$

consider 2d subsystem:

$$\left.\begin{array}{ll} \mathrm{I} & x = r\cos\varphi \quad |\ (\cdot)^2 \\ \mathrm{II} & y = r\sin\varphi \quad |\ (\cdot)^2 \end{array}\right\} (+) \implies x^2+y^2 = r^2(\cos^2\varphi+\sin^2\varphi)$$

$$\underline{\underline{r = \sqrt{x^2+y^2}}}$$

$$\frac{\mathrm{II}}{\mathrm{I}}: \frac{y}{x} = \tan\varphi \implies \underline{\underline{\varphi = \arctan\frac{y}{x}}}$$

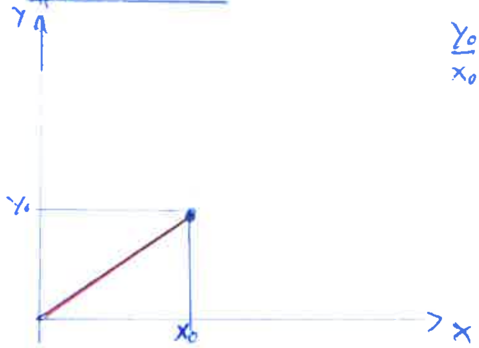


$\arctan(x): \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

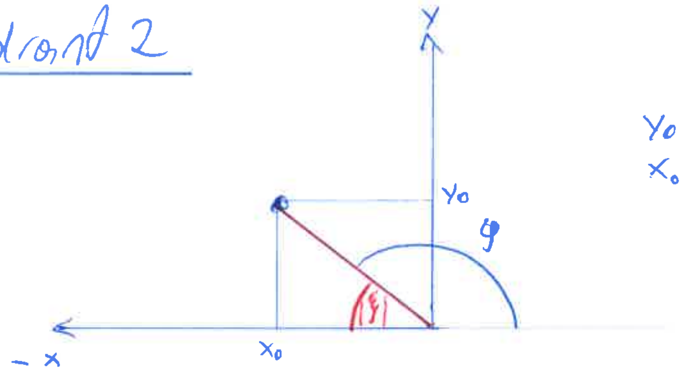
Consider quadrants:

Quadrant I:

$$\frac{y_0}{x_0} > 0,\ x_0>0,\ y_0>0 \implies \underline{\underline{\varphi = \arctan\frac{y}{x}}}$$



Quadrant 2



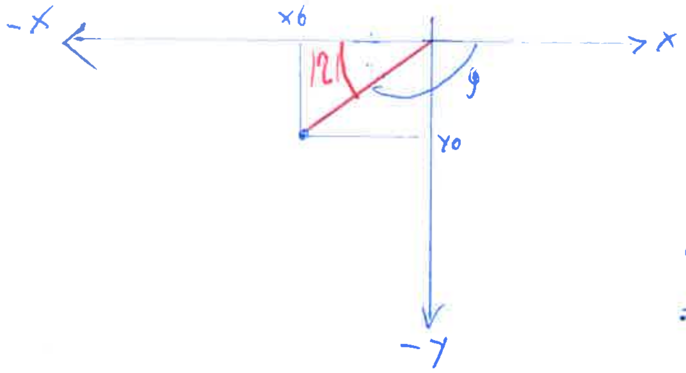
$y_0 > 0$ $\quad \frac{y_0}{x_0} < 0$

$x_0 < 0$

$\arctan \frac{y_0}{x_0} = -\xi$

$\Rightarrow \pi + \arctan \frac{y_0}{x_0} = \varphi$

Quadrant 3



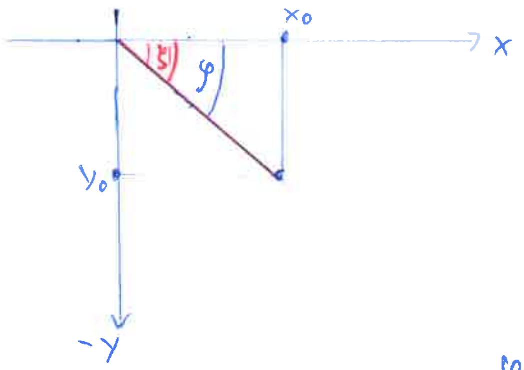
$x_0 < 0$ $\quad \frac{y_0}{x_0} > 0$

$y_0 < 0$

$\arctan \frac{y_0}{x_0} = \eta$

$\pi - \arctan \frac{y_0}{x_0} = \varphi$

Quadrant 4



$$\left.\begin{array}{l} x_0 > 0 \\ y_0 < 0 \\ \frac{y_0}{x_0} < 0 \end{array}\right] \Rightarrow \arctan \frac{y_0}{x_0} = -\xi$$

$\arctan \frac{y_0}{x_0} = \varphi$

inverted:

φ: $x = 0, y > 0$: $+\frac{\pi}{2}$ $\quad x = 0, y < 0$: $-\frac{\pi}{2}$

$y = 0, x \neq 0$: 0, $\quad y = 0, x = 0$: undef

$\Rightarrow$ $r = \sqrt{x^2 + y^2}$

$$\varphi = \begin{cases} \arctan \frac{y}{x} & x > 0, y > 0 \\ \pi + \arctan \frac{y}{x} & x < 0, y > 0 \\ \pi - \arctan \frac{y}{x} & x < 0, y < 0 \\ \arctan \frac{y}{x} & x > 0, y < 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ 0 & x \neq 0, y = 0 \\ \text{undef} & x = 0, y = 0 \end{cases}$$

$z = 0$

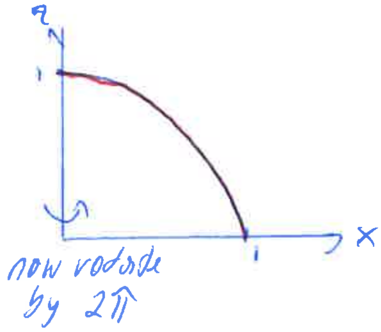
cf: programming languages:

atan2 ~ arg(z)

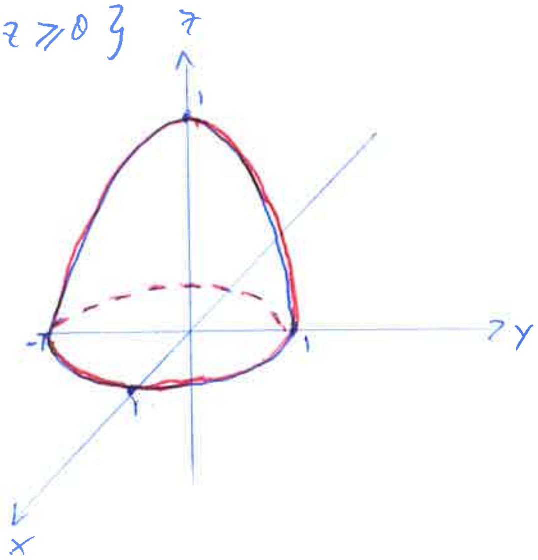
Aufgabe 4

$$S := \{ (x,y,z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2,\ z \geq 0 \}$$

<u>in z-x plane:</u>



$\rightarrow$



<u>Parametrize Area:</u>

$$\vec{r} = \begin{pmatrix} \sqrt{1-z}\cos\varphi \\ \sqrt{1-z}\sin\varphi \\ z \end{pmatrix} \qquad z \in [0,1], \quad \varphi \in [0, 2\pi)$$

$$dS = \left| \frac{\partial \vec{r}}{\partial z} \times \frac{\partial \vec{r}}{\partial \varphi} \right| dz\, d\varphi$$

$$\frac{\partial \vec{r}}{\partial z} = \begin{pmatrix} \frac{1}{2}\frac{-1}{\sqrt{1-z}}\cos\varphi \\ \frac{1}{2}\frac{-1}{\sqrt{1-z}}\sin\varphi \\ 1 \end{pmatrix} \qquad \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -\sqrt{1-z}\sin\varphi \\ \sqrt{1-z}\cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial z} \times \frac{\partial \vec{r}}{\partial \varphi} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{-\cos\varphi}{2\sqrt{1-z}} & \frac{-\sin\varphi}{2\sqrt{1-z}} & 1 \\ -\sqrt{1-z}\sin\varphi & \sqrt{1-z}\cos\varphi & 0 \end{vmatrix} = \begin{pmatrix} -\sqrt{1-z}\cos\varphi \\ -\sqrt{1-z}\sin\varphi \\ -\frac{1}{2} \end{pmatrix}$$

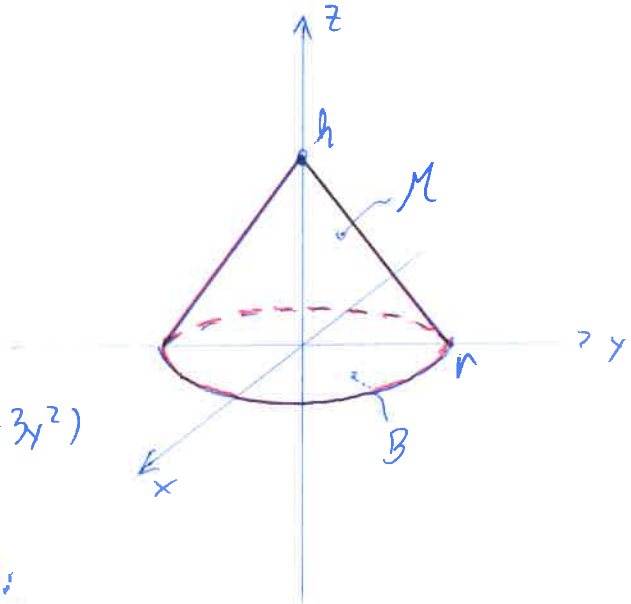
$$\Rightarrow dS = \left| \frac{\partial \vec{r}}{\partial z} \times \frac{\partial \vec{r}}{\partial \varphi} \right| dz\, d\varphi = \sqrt{(1-z) + \tfrac{1}{4}}\, dz\, d\varphi$$

$$f(x,y,z) \rightarrow f(z,\varphi) = z\sqrt{\tfrac{1}{4} + 1 - z}$$

$$\Rightarrow \int_S dS\, f(z,\varphi) = \int_0^1 dz \int_0^{2\pi} d\varphi\, z\left[\tfrac{5}{4} - z\right] = 2\pi \int_0^1 dz \left(\tfrac{5z}{4} - z^2\right)$$

$$= 2\pi\left(\tfrac{5}{8} - \tfrac{1}{3}\right) = 2\pi\left(\tfrac{7}{24}\right) = \underline{\underline{\frac{7\pi}{12}}}$$

Aufgabe 5



$(x,y,z) \in \mathbb{R}^3$

$v(x,y,z) := (z, x, -3y^2)$

Starting with (b):

$\hat{n} = -\hat{e}_z$, use polar coords (cyl. coords)

$$\int_B dA\, v \cdot \hat{n} = \left| \begin{array}{l} v \to v = (z, r\cos\varphi, -3r^2\sin^2\varphi) \\ v \cdot \hat{n} = 3r^2 \sin^2\varphi \\ \Rightarrow dA\; v \cdot \hat{n} = 3r^3 \sin^2\varphi\, dr\, d\varphi \end{array} \right|$$

$$= 3\int_0^r d\rho \int_0^{2\pi} d\varphi\, \rho^3 \sin^2\varphi = \underline{\underline{\frac{3r^4}{4}\pi}}$$

Now calculate overall Flux by Gauss' Theorem:

use cyl. coords:

$$\vec{r} = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix} \qquad \begin{array}{l} \rho \in [0, \frac{r}{h}(h-z)] \\ \varphi \in [0, 2\pi) \\ z \in [0, h] \end{array}$$

$$\Rightarrow \int_V \vec{\nabla} \cdot \vec{v}\, dV = \left| \begin{array}{l} \vec{\nabla} \cdot \vec{v} = 0 \\ \Rightarrow \int_V \vec{\nabla} \cdot \vec{v}\, dV = 0 \end{array} \right| = 0 \Rightarrow$$

(a) contribution of M: $\underline{\underline{M = -\frac{3r^4}{4}\pi}}$