Vektoranalysis (für PhysikerInnen) SS 2014

8. Übungsblatt 27. Mai

Aufgabe 1
Berechnen Sie das Flussintegral

$$\iint_{\partial W} \begin{pmatrix} x^2 + e^{y^2+z^2} \\ y^2 + x^2 z^2 \\ z^2 - e^y \end{pmatrix} dA,$$

wobei $W = [0,1] \times [0,1] \times [0,1]$ der Einheitswürfel im $\mathbb{R}^3$ ist.
Tipp: Der Satz von Gauß könnte nützlich sein.

Aufgabe 2
Es seien $\vec{F}(x,y,z) = (xy+1, x+y^2, x+z)$ und V die Pyramide mit den Eckpunkten $(0,0,0), (1,0,0)$, $(0,1,0)$ und $(0,0,1)$. Verifizieren Sie in dieser Situation den Satz von Gauß, d.h. berechnen Sie

$$\iiint_V \nabla \cdot \vec{F} dV \quad \text{und} \quad \iint_{\partial V} \vec{F} \cdot \vec{n} dA$$

und vergleichen Sie.

Aufgabe 3
Es bezeichne C den positiv orientierten Rand des Dreiecks mit den Eckpunkten $(0,0), (2,0)$ und $(4,2)$. Berechnen Sie

$$\oint_C (2x^2 + y^2)dx + (3y - 4x)dy.$$

Hinweis: Der Satz von Green in der Ebene hilft.

Aufgabe 4
Es sei $V \subset \mathbb{R}^3$ ein Gebiet, das die Voraussetzungen des Satzes von Gauß erfüllt. Zeigen Sie, dass für zweimal stetig differenzierbare Funktionen $f, g : \overline{V} \to \mathbb{R}$ die Formel

$$\iiint_V f \Delta g dV + \iiint_V \nabla f \cdot \nabla g dV = \iint_{\partial V} f \nabla g \cdot \vec{n} dA$$

gilt.

Aufgabe 5
Es seien $M \subset \mathbb{R}^2$ ein von einer geschlossenen, stückweise glatten Kurve berandetes Gebiet und $d_1, d_2 : \overline{M} \to \mathbb{R}$ stetig differenzierbare Funktionen mit $d_1(x,y) \leq d_2(x,y)$ für alle $(x,y) \in M$. Weiter sei

$$V := \left\{ \vec{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : (x_1, x_2) \in M, d_1(x_1, x_2) \leq x_3 \leq d_2(x_1, x_2) \right\} \subset \mathbb{R}^3.$$

Überprüfen Sie **ohne Zuhilfenahme des Satzes von Gauß**, dass die Identität

$$\iiint_V \frac{\partial g}{\partial x_3} dV = \iint_{\partial V} g n_3 dA$$

für jede stetig differenzierbare Funktion $g : \overline{V} \to \mathbb{R}$ gilt, wobei $\vec{n} = (n_1, n_2, n_3)^T$ das auswärts zeigende Einheitsnormalenfeld am Rand von V ist.
Ideen: Skizzieren Sie zunächst den Bereich V; finden Sie Parametrisierungen der "Deckfläche" und der "Bodenfläche" und berechnen Sie dann das Einheitsnormalenfeld $\vec{n}$; berechnen Sie damit das Integral auf der rechten Seite der Behauptung und nutzen Sie dabei den Hauptsatz der Differential- und Integralrechnung bzgl. der Variable x_3.

A. Windisch

Aufgabe 1

$$W = [0,1] \times [0,1] \times [0,1] \in \mathbb{R}^3$$

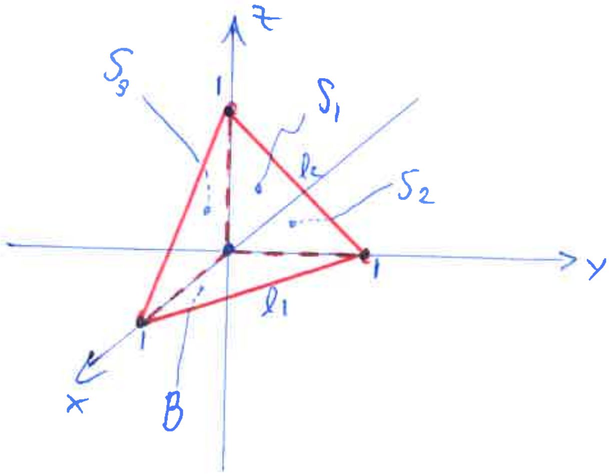
Gauss:
$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial V} \vec{F}\cdot d\vec{A}$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} x^2 + e^{y^2+z^2} \\ y^2 + x^2 z^2 \\ z^2 - e^{y} \end{pmatrix} = 2x + 2y + 2z$$

$$\Rightarrow \quad 2\int_0^1 dx \int_0^1 dy \int_0^1 dz\,(x+y+z) = 2\left(\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}\right) = \underline{\underline{3}}$$

Aufgabe 2

$$\vec{F} = \begin{pmatrix} xy+1 \\ x+y^2 \\ x+z \end{pmatrix}$$



show that

$$\iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial V} \vec{F}\cdot\hat{n}\, dA$$

Parametrize boundaries for volume integration:

Start by parametrizing the base area:
Use x as parameter, $x \in [0,1] \Rightarrow y = 1 - x$

With increasing z, this area gets smaller.
The parametrisation for z reads

$z = 1-x-y$, hence the overall integral becomes

$$\iiint_V dV = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = \int_0^1 dx \int_0^{1-x} dy\,(1-x-y)$$

$$= \int_0^1 dx \left[1-x - \frac{(1-x)^2}{2} - x(1-x) \right]$$

$$= \int_0^1 dx \left[1 \cancel{-x} - \frac{1}{2} - \frac{x^2}{2} \cancel{+x} - x + x^2 \right]$$

$$= \int_0^1 dx \left[\frac{1}{2} - x + \frac{x^2}{2} \right] = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \underline{\underline{\frac{1}{6}}}$$

<u>Now we can evaluate the lhs.:</u>

$$\vec{\nabla}\cdot\vec{F} = [y + 2y + 1] = \underline{\underline{3y+1}}$$

$$\Rightarrow \iiint_V dV\, \vec{\nabla}\cdot\vec{F} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\,(3y+1) = \int_0^1 dx \int_0^{1-x} dy\,(3y+1)(1-x-y)$$

$$= \int_0^1 dx \int_0^{1-x} dy\,[3y - 3xy - 3y^2 + 1 - x - y] = \int_0^1 dx \int_0^{1-x} dy\,[1 - x + 2y - 3xy - 3y^2]$$

$$= \int_0^1 dx \left[(1-x)^2 + \frac{(1-x)^2}{2}(2-3x) - (1-x)^3 \right]$$

$$= \int_0^1 dx \left[1 - 2x + x^2 + \frac{1}{2}(1-2x+x^2)(2-3x) - (1-x)(1-2x+x^2) \right]$$

$$= \int_0^1 dx \left[1 - 2x + x^2 \cancel{+1} - 2x + x^2 - \frac{3}{2}x + 3x^2 - \frac{3x^3}{2} \cancel{-1} + 2x - x^2 + x - 2x^2 + x^3 \right]$$

$$= \int_0^1 dx \left[1 - \frac{5}{2}x + 2x^2 - \frac{x^3}{2} \right] = 1 - \frac{5}{4} + \frac{2}{3} - \frac{1}{8} = (24 - 30 + 16 - 3)\frac{1}{24} = \underline{\frac{7}{24}}$$

For the rhs we need the unit vector for the faces:

(see Sketch).

base B: $\hat{n} = -\hat{e}_z, \; z = 0$

$$\Rightarrow B = \int_0^1 dx \int_0^{1-x} dy\, \vec{F}\cdot(-\hat{e}_z) = -\int_0^1 dx \int_0^{1-x} dy\, x = -\int_0^1 dx\, x(1-x)$$

$$= \left(-\frac{x^2}{2} + \frac{x^3}{3}\right)\Big|_0^1 = -\frac{1}{2} + \frac{1}{3} = \underline{\underline{-\frac{1}{6}}}$$

face S_1: construct area element:

the face S_1 is the set of all points x, y, z:

$$S_1 = \{(x,y,z) \in \mathbb{R}^3 : x + y + z = 1,\ x > 0,\ y > 0,\ z > 0\}$$

$\Rightarrow$ use x, y and $z = 1 - x - y$

$$\Rightarrow \vec{r} = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix} \qquad dS = \left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| dx\, dy$$

$$\left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\| = \left\| \begin{vmatrix} e_x & e_y & e_z \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{3}, \quad \Rightarrow dS = \sqrt{3}\, dx\, dy$$

$$\text{and} \quad \hat{n} = \frac{\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}}{\left\| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right\|} = \underline{\underline{\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}}$$

The integrand becomes $\hat{n}\cdot\vec{F} = \frac{1}{\sqrt{3}}(xy + 1 + x + y^2 + \cancel{x} + 1 - \cancel{x} - y)$,

and we get

$$S_1 = \underbrace{\int_0^1 dx \int_0^{1-x} dy\, \cancel{\sqrt{3}}}_{dS}\ \underbrace{\frac{1}{\cancel{\sqrt{3}}}(2 + x - y + xy + y^2)}_{\hat{n}\cdot F} =$$

$$= \int_0^1 dx \left[2(1-x) + x(1-x) - \frac{(1-x)^2}{2} + x\frac{(1-x)^2}{2} + \frac{(1-x)^3}{3} \right] = \%$$

$$= \int_0^1 dx \left[2 - 2x + x - x^2 - \frac{1}{2} - \frac{x^2}{2} + x + \frac{x}{2} - x^2 + \frac{x^3}{2} + \left(\frac{1}{3} - \frac{x}{3}\right)(1 - 2x + x^2) \right] \quad (4)$$

$$= \int_0^1 dx \left[2 - 2x + \cancel{x} - x^2 - \frac{1}{2} - \frac{x^2}{2} + x + \frac{x}{2} - \cancel{x^2} + \frac{x^3}{2} + \frac{1}{3} - \cancel{\frac{2x}{3}} + \cancel{\frac{x^2}{3}} - \cancel{\frac{x}{3}} + \cancel{\frac{2x^2}{3}} - \frac{x^3}{3} \right]$$

$$= \int_0^1 dx \left[\frac{11}{6} - \frac{x}{2} - \frac{3x^2}{2} + \frac{x^3}{6} \right]$$

$$= \frac{11}{6} - \frac{1}{4} - \frac{1}{2} + \frac{1}{24} = \frac{1}{24}\left[44 - 6 - 12 + 1\right] = \frac{27}{24} = \underline{\underline{\frac{9}{8}}}$$

<u>face S_2:</u> $\hat{n} = -\hat{e}_x , \quad x = 0$

$$\Rightarrow \quad S_2 = \int_0^1 dz \int_0^{1-z} dy \; \vec{F} \cdot (-\hat{e}_x) = -\int_0^1 dz \int_0^{1-z} dy = -\int_0^1 dz \, (1 - z) = -1 + \frac{1}{2} = \underline{\underline{-\frac{1}{2}}}$$

<u>face S_3.</u> $\hat{n} = -\hat{e}_y , \quad y = 0$

$$\Rightarrow \quad S_3 = \int_0^1 dz \int_0^{1-z} dx \; \vec{F} \cdot (-\hat{e}_y) = -\int_0^1 dz \int_0^{1-z} dx \, x = B = \underline{\underline{-\frac{1}{6}}}$$

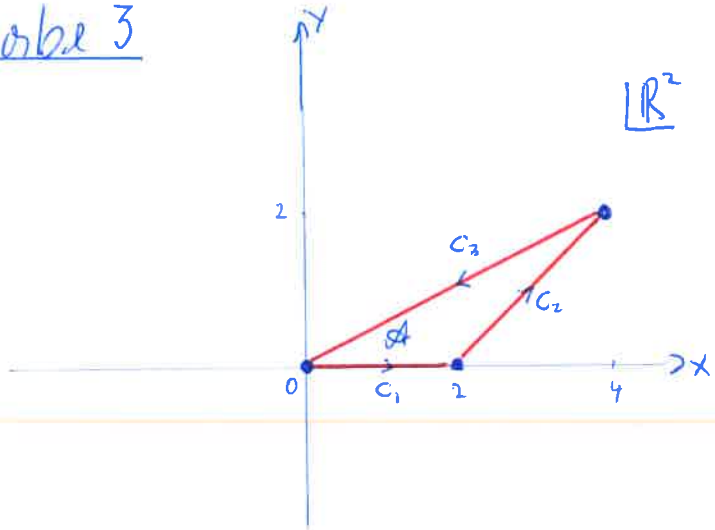
$\Rightarrow$ <u>rhs:</u>

$$\iint_{\partial V} \vec{F} \cdot \hat{n} \, dA = B + S_1 + S_2 + S_3 = -\frac{1}{6} - \frac{1}{2} - \frac{1}{6} + \frac{9}{8}$$

$$= \frac{1}{24}(-4 - 12 - 4 + 27) = \underline{\underline{\frac{7}{24}}}$$

$\Rightarrow$ $\underline{\underline{\text{lhs} = \text{rhs}}}$

Aufgabe 3



$$\partial A = C = C_1 + C_2 + C_3$$

Green's theorem in the plane:

$$\iint_A \left(\frac{\partial g(x,y)}{\partial x} - \frac{\partial f(x,y)}{\partial y}\right) dx\,dy = \oint_{\partial A} \left(f(x,y)dx + g(x,y)dy\right)$$

Parametrization of A:

$y \in [0,2]$

$x_{\text{upper boundary}} = 2 + y$

$x_{\text{lower boundary}} = 2y$

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial}{\partial x}(3y - 4x) = \underline{\underline{-4}}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}(2x^2 + y^2) = \underline{\underline{2y}}$$

$$-\int_0^2 dy \int_{2y}^{2+y} dx\, 4 - 2\int_0^2 dy \int_{2y}^{2+y} dx\, y = \int_0^2 dy\left[-4(2+y-2y)\right] + \int_0^2 dy\left[-2y(2+y-2y)\right]$$

$$= \int_0^2 dy\left[-8 - \cancel{4y} + \cancel{8y} - \cancel{4y} - 2y^2 + 4y^2\right] = \int_0^2 dy\left[-8 + 2y^2\right]$$

$$= -16 + \frac{2}{3}\cdot 8 = -16 + \frac{16}{3} = \underline{\underline{-\frac{32}{3}}}$$

Let us check this result by integrating along the boundary of A, i.e. along the contour $C = C_1 + C_2 + C_3$. We have

$$\underbrace{\int_0^2 dx(2x^2)}_{C_1} + \underbrace{\int_2^4 dx\left(2x^2 + (x-2)^2\right) + \int_0^2 dy\left(3y - 4(y+2)\right)}_{(y = x-2)\quad C_2 \quad (x = y+2)} - \underbrace{\int_0^4 dx\left(2x^2 + \frac{x^2}{4}\right) - \int_0^2 dy(3y - 8y)}_{(y=\frac{x}{2})\quad C_3 \quad (x=2y)} =$$

Here I flipped the sign because of the orientation of the segment

$= \%$

⑥

$$\% = 2\cdot\frac{8}{3} + \int_2^4 dx\,[2x^2 + x^2 + 4 - 4x] + \int_0^2 dy\,[-y-8] - \int_0^4 dx\,[x^2(\tfrac{9}{4})] + \int_0^2 dy\,[5y]$$

$$= \frac{16}{3} + \int_2^4 dx\,[3x^2 - 4x + 4] - 2 - 16 - \frac{\cancel{9}^{\,3}}{\cancel{4}}\cdot\frac{1}{\cancel{3}}\cancel{64}^{\,16} + 5\cdot 2$$

$$= \frac{16}{3} - 18 - 48 + 10 \underbrace{+64 \cancel{-8} - 32 + 8}_{=40} \cancel{+16} \cancel{-8}$$

$$= \frac{16}{3} - 8 - 8 = \frac{16}{3} - \frac{48}{3} = \underline{\underline{-\frac{32}{3}}},$$ which equals the previous result.

Aufgabe 4

Show, that $$\iiint_V f\Delta g\, dV + \iiint_V \nabla f\cdot\nabla g\, dV = \iint_{\partial V} f\nabla g\cdot\hat{n}\, dA$$

define $\vec{X} := f\vec{\nabla} g$, such that the rhs of the above equation reads

$$\iint_{\partial V} \vec{X}\cdot\hat{n}\, dA.$$

According to Gauss' theorem this equals

$$\iint_{\partial V} \vec{X}\cdot\hat{n}\, dA = \iiint_V \vec{\nabla}\cdot\vec{X}\, dV,$$

thus the above relation holds if $\vec{\nabla}\cdot\vec{X} = f\Delta g + \vec{\nabla} f\cdot\vec{\nabla} g$ and if f, g are at least C^2 functions. Indeed we find that

$$\vec{\nabla}\cdot\vec{X} = \vec{\nabla}\cdot(f\vec{\nabla} g) = (\vec{\nabla} f)\cdot(\vec{\nabla} g) + f\Delta g,$$

which proves the relation.

□

Aufgabe 5

$$M \subset \mathbb{R}^2$$

$d_1, d_2 : \overline{M} \to \mathbb{R}$, $d_1(x,y) \leq d_2(x,y) \quad \forall (x,y) \in M$

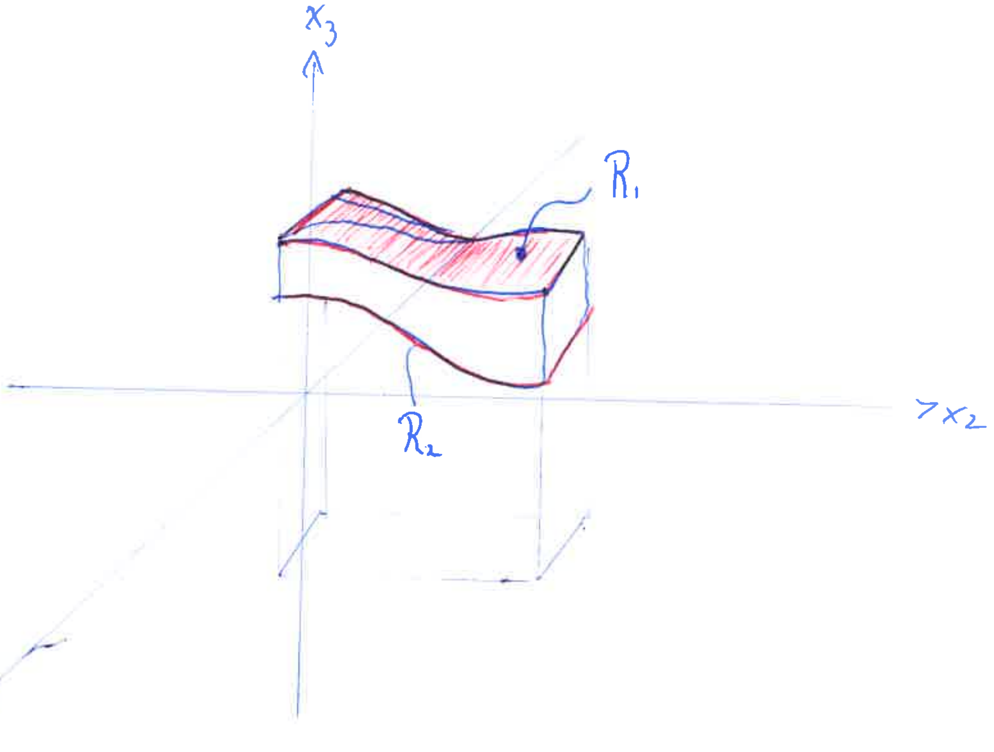
$$V = \{\vec{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : (x_1, x_2) \in M, d_1(x_1,x_2) \leq x_3 \leq d_2(x_1,x_2)\} \subset \mathbb{R}^3$$

Show, that

$$\iiint_V \frac{\partial f}{\partial x_3} \, dV = \iint_{\partial V} f n_3 \, dA$$

for all $f : \overline{V} \to \mathbb{R}$ continuous and differentiable, and $\vec{n}$ is the outward pointing area unit vector $\vec{n} = (n_1, n_2, n_3)^T$.

Sketch:



Parametrization of boundaries R_i, $i \in \{1,2\}$:

$$\vec{P}_i(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ d_i(x_1, x_2) \end{pmatrix}$$

Now we can calculate the unit vectors of the boundaries.

$$\frac{\partial}{\partial x_1}\vec{P}_i(x_1,x_2) \times \frac{\partial}{\partial x_2}\vec{P}_i(x_1,x_2) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & \frac{\partial}{\partial x_1} d_i \\ 0 & 1 & \frac{\partial}{\partial x_2} d_i \end{vmatrix} = \begin{pmatrix} -\frac{\partial}{\partial x_1} d_i \\ -\frac{\partial}{\partial x_2} d_i \\ 1 \end{pmatrix}$$

Now we have to normalize to unit length, which gives

$$\hat{n}_i = \frac{(-1)^{i+1}}{\sqrt{\left(\frac{\partial}{\partial x_1} d_i\right)^2 + \left(\frac{\partial}{\partial x_2} d_i\right)^2 + 1}} \begin{pmatrix} -\frac{\partial}{\partial x_1} d_i \\ -\frac{\partial}{\partial x_2} d_i \\ 1 \end{pmatrix}$$

, where we introduced an additional sign to ensure that the unit vector of the lower surface points downwards.

The scalar surface element for the boundaries R_i is then given by

$$\begin{aligned} d\sigma_i &= \left\| \frac{\partial}{\partial x_1}\vec{P}_i \times \frac{\partial}{\partial x_2}\vec{P}_i \right\| dx_1\, dx_2 \\ &= \sqrt{\left(\frac{\partial}{\partial x_1} d_i\right)^2 + \left(\frac{\partial}{\partial x_2} d_i\right)^2 + 1}\; dx_1\, dx_2 \,. \end{aligned}$$

Now we can evaluate the rhs of the assertion,

$$\begin{aligned} \iint_{\partial V} d\sigma\, g\, n_3 &= \int_{R_1} d\sigma_1 (\hat{n}_1)_3\, g + \int_{R_2} d\sigma_2 (\hat{n}_2)_3\, g \\ &= \int_M dx_1\, dx_2 \frac{\cancel{\sqrt{\left(\frac{\partial}{\partial x_1} d_1\right)^2 + \left(\frac{\partial}{\partial x_2} d_1\right)^2 + 1}}}{\cancel{\sqrt{\left(\frac{\partial}{\partial x_1} d_1\right)^2 + \left(\frac{\partial}{\partial x_2} d_1\right)^2 + 1}}} \cdot 1 \cdot g(x_1, x_2, d_1(x_1,x_2)) \\ &+ \int_M dx_1\, dx_2 \frac{\cancel{\sqrt{\left(\frac{\partial}{\partial x_1} d_2\right)^2 + \left(\frac{\partial}{\partial x_2} d_2\right)^2 + 1}}}{\cancel{\sqrt{\left(\frac{\partial}{\partial x_1} d_2\right)^2 + \left(\frac{\partial}{\partial x_2} d_2\right)^2 + 1}}} \cdot (-1)\, g(x_1, x_2, d_2(x_1,x_2)) = \% \end{aligned}$$

$$\% = \int_M dx_1\, dx_2\, g(x_1, x_2, d_1(x_1,x_2)) - \int_M dx_1\, dx_2\, g(x_1, x_2, d_2(x_1,x_2))$$

$$= \Big| \text{use fundamental theorem of calculus} \Big|$$

$$= \int_M dx_1\, dx_2 \int_{d_2(x_1,x_2)}^{d_1(x_1,x_2)} \partial_{x_3}\, g(x_1, x_2, x_3) = \underline{\underline{\int_V \partial_{x_3} g}},$$

which proves the assertion. $\square$