Uni Graz
TU Graz

S. Kofler, A. Windisch
M. Holzmann, J. Rohleder, M. Zank

Vektoranalysis (für PhysikerInnen)

SS 2014

7. Übungsblatt

20. Mai

Aufgabe 1
Zeigen Sie, dass die Rechenregel

$$\nabla \times (\vec{F} \times \vec{G}) = (\nabla \cdot \vec{G})\vec{F} - (\nabla \cdot \vec{F})\vec{G} + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$$

für alle (hinreichend oft differenzierbaren) Vektorfelder $\vec{F}$ und $\vec{G}$ gilt.

Aufgabe 2
Gegeben seien elliptische Zylinderkoordinaten

$$\vec{r} = \begin{pmatrix} x_1(u,v,z) \\ x_2(u,v,z) \\ x_3(u,v,z) \end{pmatrix} = \begin{pmatrix} a\cosh(u)\cos(v) \\ a\sinh(u)\sin(v) \\ z \end{pmatrix}$$

mit festem $a > 0$.

(i) Zeichnen Sie die Koordinatenlinien in der Ebene $z = 0$.

(ii) Berechnen Sie die Basisvektoren $\vec{e}_u, \vec{e}_v, \vec{e}_z$ und zeigen Sie deren Orthogonalität.

(iii) Bestimmen Sie die Komponenten des Ortsvektors $\vec{r}$ in der obigen Basis und berechnen Sie für $v = \pi/2 = const$ den Geschwindigkeitsvektor $\dot{\vec{r}}(t)$ in dieser Basis.

Aufgabe 3
Gegeben seien parabolische Koordinaten

$$\vec{r} = \begin{pmatrix} x_1(u,v,\varphi) \\ x_2(u,v,\varphi) \\ x_3(u,v,\varphi) \end{pmatrix} = \begin{pmatrix} uv\cos\varphi \\ uv\sin\varphi \\ \frac{1}{2}(u^2 - v^2) \end{pmatrix}.$$

(i) Bestimmen Sie die Basisvektoren $\vec{e}_u, \vec{e}_v, \vec{e}_\varphi$ und zeigen Sie ihre Orthogonalität.

(ii) Berechnen Sie den Nabla-Operator in obiger Basis und bestimmen Sie den Gradienten des Feldes

$$\Phi(u,v,\varphi) = u^2 + v^2 - uv.$$

Aufgabe 4
Drücken Sie den Vektor $\vec{a} = x_3\vec{e}_1 - 2x_1\vec{e}_2 + x_2\vec{e}_3$ in Zylinderkoordinaten aus, d.h. in den entsprechenden Variablen ρ, φ, z und Einheitsvektoren $\vec{e}_\rho, \vec{e}_\varphi, \vec{e}_z$.

Lösungen zum 7. Übungsblatt aus Vektoranalysis SS14

A. Windisch

Aufgabe 1

show that $\vec{\nabla} \times (\vec{F} \times \vec{G}) = (\vec{\nabla} \cdot \vec{G})\vec{F} - (\vec{\nabla} \cdot \vec{F})\vec{G} + (\vec{G} \cdot \vec{\nabla})\vec{F} - (\vec{F} \cdot \vec{\nabla})\vec{G}$

holds.

lhs:

$$\vec{F} \times \vec{G} = \begin{vmatrix} e_1 & e_2 & e_3 \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = \begin{pmatrix} F_2 G_3 - F_3 G_2 \\ F_3 G_1 - F_1 G_3 \\ F_1 G_2 - F_2 G_1 \end{pmatrix}$$

$$\vec{\nabla} \times \vec{F} \times \vec{G} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_2G_3 - F_3G_2 & F_3G_1 - F_1G_3 & F_1G_2 - F_2G_1 \end{vmatrix} = \begin{pmatrix} \partial_2(F_1G_2 - F_2G_1) - \partial_3(F_3G_1 - F_1G_3) \\ \partial_3(F_2G_3 - F_3G_2) - \partial_1(F_1G_2 - F_2G_1) \\ \partial_1(F_3G_1 - F_1G_3) - \partial_2(F_2G_3 - F_3G_2) \end{pmatrix}$$

rhs:

term 1:

$$(\vec{\nabla} \cdot \vec{G})\vec{F} = \begin{pmatrix} (\partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3) F_1 \\ (\partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3) F_2 \\ (\partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3) F_3 \end{pmatrix}$$

term 2:

$$(\vec{\nabla} \cdot \vec{F})\vec{G} = \begin{pmatrix} (\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) G_1 \\ (\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) G_2 \\ (\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) G_3 \end{pmatrix}$$

term 3:

$$(\vec{G} \cdot \vec{\nabla})\vec{F} = \begin{pmatrix} (G_1 \partial_1 + G_2 \partial_2 + G_3 \partial_3) F_1 \\ (G_1 \partial_1 + G_2 \partial_2 + G_3 \partial_3) F_2 \\ (G_1 \partial_1 + G_2 \partial_2 + G_3 \partial_3) F_3 \end{pmatrix}$$

term 4:

$$(\vec{F} \cdot \vec{\nabla})\vec{G} = \begin{pmatrix} (F_1 \partial_1 + F_2 \partial_2 + F_3 \partial_3) G_1 \\ (F_1 \partial_1 + F_2 \partial_2 + F_3 \partial_3) G_2 \\ (F_1 \partial_1 + F_2 \partial_2 + F_3 \partial_3) G_3 \end{pmatrix}$$

consider 1st component of rhs vector:

$$F_1 \cancel{(\partial_1 G_1)} + F_1 \partial_2 G_2 + F_1 \partial_3 G_3 - \cancel{G_1 \partial_1 F_1} - G_1 \partial_2 F_2 - G_1 \partial_3 F_3$$

$$+ \cancel{G_1 \partial_1 F_1} + G_2 \partial_2 F_1 + G_3 \partial_3 F_1 - \cancel{F_1 \partial_1 G_1} - F_2 \partial_2 G_1 - F_3 \partial_3 G_1$$

$$= \underbrace{F_1 \partial_2 G_2 + G_2 \partial_2 F_1} - \underbrace{G_1 \partial_2 F_2 - F_2 \partial_2 G_1} \underbrace{- G_1 \partial_3 F_3 - F_3 \partial_3 G_1} + G_3 \partial_3 F_1 + F_1 \partial_3 G_3$$

$$= \partial_2 (F_1 G_2) - \partial_2 (F_2 G_1) - \partial_3 (F_3 G_1) + \partial_3 (F_1 G_3)$$

$$= \partial_2 (F_1 G_2 - F_2 G_1) - \partial_3 (F_3 G_1 - F_1 G_3)$$

consider 2nd component of rhs vector:

$$F_2 \partial_1 G_1 + \cancel{F_2 \partial_2 G_2} + F_2 \partial_3 G_3 - G_2 \partial_1 F_1 - \cancel{G_2 \partial_2 F_2} - G_2 \partial_3 F_3$$

$$+ G_1 \partial_1 F_2 + \cancel{G_2 \partial_2 F_2} + G_3 \partial_3 F_2 - F_1 \partial_1 G_2 - \cancel{F_2 \partial_2 G_2} - F_3 \partial_3 G_2$$

$$= \underbrace{F_2 \partial_3 G_3 + G_3 \partial_3 F_2} - \underbrace{G_2 \partial_3 F_3 - F_3 \partial_3 G_2} - \underbrace{G_2 \partial_1 F_1 - F_1 \partial_1 G_2} + \underbrace{G_1 \partial_1 F_2 + F_2 \partial_1 G_1}$$

$$= \partial_3 (F_2 G_3) - (\partial_3 F_3 G_2) - \partial_1 (F_1 G_2) + \partial_1 (F_2 G_1)$$

$$= \partial_3 (F_2 G_3 - F_3 G_2) - \partial_1 (F_1 G_2 - F_2 G_1)$$

consider 3rd component of rhs vector:

$$F_3 \partial_1 G_1 + F_3 \partial_2 G_2 + \cancel{F_3 \partial_3 G_3} - G_3 \partial_1 F_1 - G_3 \partial_2 F_2 - \cancel{G_3 \partial_3 F_3}$$

$$+ G_1 \partial_1 F_3 + G_2 \partial_2 F_3 + \cancel{G_3 \partial_3 F_3} - F_1 \partial_1 G_3 - F_2 \partial_2 G_3 - \cancel{F_3 \partial_3 G_3}$$

$$= \underbrace{G_1 \partial_1 F_3 + F_3 \partial_1 G_1} - \underbrace{G_3 \partial_1 F_1 - F_1 \partial_1 G_3} - \underbrace{G_3 \partial_2 F_2 - F_2 \partial_2 G_3} + \underbrace{G_2 \partial_2 F_3 + F_3 \partial_2 G_2}$$

$$= \partial_1 (F_3 G_1) - \partial_1 (F_1 G_3) - \partial_2 (F_2 G_3) + \partial_2 (F_3 G_2)$$

$$= \partial_1 (F_3 G_1 - F_1 G_3) - \partial_2 (F_2 G_3 - F_3 G_2)$$

$\Rightarrow$ lhs = rhs

Alternatively: (einstein sum convention implied)

lhs: $$[\vec{\nabla} \times (\vec{F} \times \vec{G})]_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} F_l G_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j F_l G_m = \partial_m (F_i G_m) - \partial_l (F_l G_i)$$

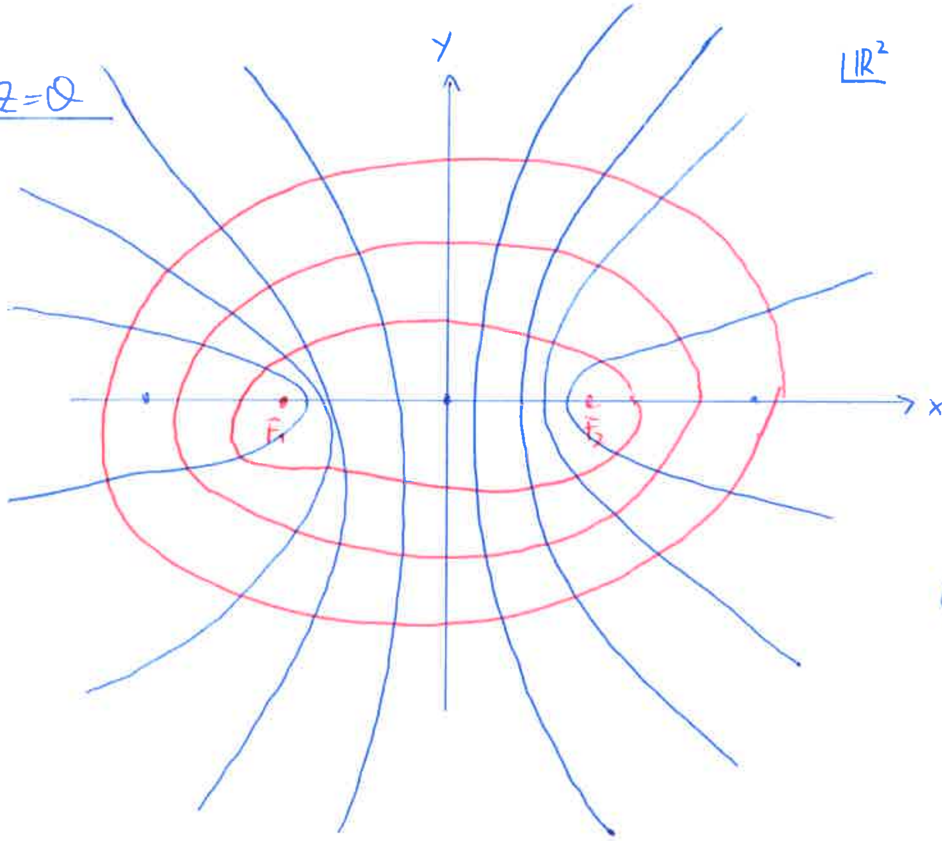
rhs: $$F_i \partial_j G_j - G_i \partial_j F_j + G_j \partial_j F_i - F_j \partial_j G_i$$

$$= \partial_j (F_i G_j) - \partial_j (F_j G_i) \Rightarrow \text{lhs.} = \text{rhs}$$

Aufgabe 2

$$\vec{r} = \begin{pmatrix} x_1(u,v,z) \\ x_2(u,v,z) \\ x_3(u,v,z) \end{pmatrix} = \begin{pmatrix} a\cosh u \cos v \\ a \sinh u \sin v \\ z \end{pmatrix} \qquad a>0$$

(i) $z=0$



$a=1$ (w.l.o.g.)

keeping v fixed while varying u yields hyperbolas with foci ± 1.

Keeping u fixed while varying v yields ellipses with foci ± 1.

(see mathematica file)

(ii) basis vectors: use tangent vectors to obtain orthonormal basis.

$$\hat{e}_1 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial u}\right\|}\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} a\sinh u \cos v \\ a\cosh u \sin v \\ 0 \end{pmatrix} \frac{1}{[a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v]^{\frac{1}{2}}}$$

$$= \begin{pmatrix} \sinh u \cos v \\ \cosh u \sin v \\ 0 \end{pmatrix} \frac{1}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}}$$

$$\hat{e}_2 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial v}\right\|}\frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} -a\cosh u \sin v \\ a\sinh u \cos v \\ 0 \end{pmatrix} \frac{1}{[a^2 \cosh^2 u \sin^2 v + a^2 \sinh^2 u \cos^2 v]^{\frac{1}{2}}} = \%$$

$$\% = \begin{pmatrix} -\cosh u \sin v \\ \sinh u \cos v \\ 0 \end{pmatrix} \frac{1}{[\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v]^{1/2}}$$

$$\hat{e}_3 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial z}\right\|} \frac{\partial \vec{r}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

<u>orthonormality:</u>

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$\hat{e}_1 \cdot \hat{e}_1 = \frac{1}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v} [\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v] = \underline{\underline{1}}$$

$$\hat{e}_1 \cdot \hat{e}_2 = \frac{1}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v} [-\sinh u \cos v \cosh u \sin v + \sinh u \cos v \cosh u \sin v]$$

$$= \underline{\underline{0}}$$

$$\hat{e}_1 \cdot \hat{e}_3 = \underline{\underline{0}}$$

$$\hat{e}_2 \cdot \hat{e}_2 = \frac{1}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v} [\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v] = \underline{\underline{1}}$$

$$\hat{e}_2 \cdot \hat{e}_3 = \underline{\underline{0}}$$

$$\hat{e}_3 \cdot \hat{e}_3 = \underline{\underline{1}}$$

$\Rightarrow$ <u>orthonormal basis.</u>

(iii) <u>coordinate vector</u>

<u>project components:</u>

<u>$\hat{e}_1$:</u> $$\vec{r} \cdot \hat{e}_1 = \begin{pmatrix} a \cosh u \cos v \\ a \sinh u \sin v \\ z \end{pmatrix} \cdot \begin{pmatrix} \sinh u \cos v \\ \cosh u \sin v \\ 0 \end{pmatrix} \frac{1}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}} = \%$$

$$\% = \frac{a \cosh u \sinh u \cos^2 v + a \sinh u \cosh u \sin^2 v}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}}$$

$$= \underline{\underline{\frac{a \sinh u \cosh u}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}}}}$$

$\hat{e}_2$: $$\vec{r}, \hat{e}_2 = \begin{pmatrix} a \cosh u \cos v \\ a \sinh u \sin v \\ z \end{pmatrix} \cdot \begin{pmatrix} -\cosh u \sin v \\ \sinh u \cos v \\ 0 \end{pmatrix} \frac{1}{[\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v]^{\frac{1}{2}}}$$

$$= \frac{-a \cosh^2 u \cos v \sin v + a \sinh^2 u \sin v \cos v}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}} = \left| \cosh^2 x - \sinh^2 x = 1 \right|$$

$$= \underline{\underline{\frac{-a \sin v \cos v}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}}}}$$

$\hat{e}_3$: $\vec{r}, \vec{e}_3 = \underline{\underline{z}}$

$$\Rightarrow \quad \vec{r}_{\text{elliptic basis}} = \underline{\underline{\begin{pmatrix} \frac{a \sinh u \cosh u}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}} \\ \frac{-a \sin v \cos v}{[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v]^{\frac{1}{2}}} \\ z \end{pmatrix}}}$$

(iii) <u>velocity vector:</u> $\dot{\vec{r}}(t)$, $u(t)$, $v = \text{const.}$, $z(t)$

$$v(t) = \dot{\vec{r}}(t) = \dot{r}_1 \hat{e}_1 + r_1 \dot{\hat{e}}_1 + \dot{r}_2 \hat{e}_2 + r_2 \dot{\hat{e}}_2 + \dot{r}_3 \hat{e}_3 + r_3 \dot{\hat{e}}_3$$

<u>let us determine the ingredients first:</u> $\underline{\underline{v = \frac{\pi}{2}}}$!

$\dot{r}_1 = \frac{d}{dt}(a \sinh u) = \underline{\underline{a \cosh u \, \dot{u}}}$ $\qquad$ $\dot{r}_2 = \frac{d}{dt}(0) = \underline{\underline{0}}$ $\qquad$ $\underline{\underline{\hat{e}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}} \Rightarrow \underline{\underline{\dot{\hat{e}}_2 = \vec{0}}}$

$\underline{\underline{\hat{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}} \Rightarrow \underline{\underline{\dot{\hat{e}}_1 = \vec{0}}}$ $\qquad$ $\underline{\underline{\dot{r}_3 = \dot{z}}}$ $\qquad$ $\underline{\underline{\hat{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}} \Rightarrow \underline{\underline{\dot{\hat{e}}_3 = 0}}$

$\vartheta = \frac{\pi}{2} = const:$

$$\Rightarrow \quad v(t) = \dot{\vec{r}}(t) = \dot{r}_1 \hat{e}_1 + \cancel{r_1 \underset{0}{\dot{\hat{e}}_1}} + \cancel{\underset{0}{\dot{r}_2}} \hat{e}_2 + \cancel{r_2 \underset{0}{\dot{\hat{e}}_2}} + \dot{r}_3 \hat{e}_3 + \cancel{r_3 \underset{0}{\dot{\hat{e}}_3}}$$

$$= \dot{r}_1 \hat{e}_1 + \dot{r}_3 \hat{e}_3$$

$$= \underline{\underline{a \dot{u} \cos u \, \hat{e}_y + \dot{z} \, \hat{e}_z}}$$

Aufgabe 3

$$\vec{r} = \begin{pmatrix} x_1(u, v, \varphi) \\ x_2(u, v, \varphi) \\ x_3(u, v, \varphi) \end{pmatrix} = \begin{pmatrix} u v \cos\varphi \\ u v \sin\varphi \\ \frac{1}{2}(u^2 - v^2) \end{pmatrix}$$

(i) Basis vectors

$$\hat{e}_1 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial u}\right\|} \frac{\partial \vec{r}}{\partial u} = \underline{\underline{\begin{pmatrix} v \cos\varphi \\ v \sin\varphi \\ u \end{pmatrix} \frac{1}{[u^2 + v^2]^{\frac{1}{2}}}}}$$

$$\hat{e}_2 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial v}\right\|} \frac{\partial \vec{r}}{\partial v} = \underline{\underline{\begin{pmatrix} u \cos\varphi \\ u \sin\varphi \\ -v \end{pmatrix} \frac{1}{[u^2 + v^2]^{\frac{1}{2}}}}}$$

$$\hat{e}_3 = \frac{1}{\left\|\frac{\partial \vec{r}}{\partial \varphi}\right\|} \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -\cancel{uv} \sin\varphi \\ \cancel{uv} \cos\varphi \\ 0 \end{pmatrix} \frac{1}{\cancel{uv}} = \underline{\underline{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}}$$

Orthonormality:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

$$\hat{e}_1 \cdot \hat{e}_1 = \frac{1}{u^2 + v^2}(u^2 + v^2) = \underline{\underline{1}}$$

$$\hat{e}_1 \cdot \hat{e}_2 = \frac{1}{u^2 + v^2}(vu - vu) = \underline{\underline{0}}$$

$$\hat{e}_1 \cdot \hat{e}_3 = \frac{1}{[u^2 + v^2]^{\frac{1}{2}}}(-v \sin\varphi \cos\varphi + v \sin\varphi \cos\varphi) = \underline{\underline{0}}$$

$$\hat{e}_2 \cdot \hat{e}_2 = \frac{1}{u^2+v^2}(u^2+v^2) = \underline{\underline{1}}$$

$$\hat{e}_2 \cdot \hat{e}_3 = \frac{1}{[u^2+v^2]^{\frac{1}{2}}}(-u \sin\varphi \cos\varphi + u \sin\varphi \cos\varphi) = \underline{\underline{0}}$$

$$\hat{e}_3 \cdot \hat{e}_3 = \sin^2\varphi + \cos^2\varphi = \underline{1} \quad \Rightarrow \quad \underline{\text{orthonormal basis.}}$$

(iii) determine $\vec{\nabla}\phi$ in this basis

consider some scalar field $\phi(x_1, x_2, x_3)$:
The u_i-th component of $\vec{\nabla}\phi$ reads:

$$(\vec{\nabla}\phi)_{u_i} = \vec{\nabla}\phi(x_1(u_1,u_2,u_3), x_2(u_1,u_2,u_3), x_3(u_1,u_2,u_3)) \cdot \hat{e}_{u_i}$$

$$= \vec{\nabla}\phi(u_1,u_2,u_3)\frac{1}{\|\frac{\partial\vec{r}}{\partial u_i}\|}\frac{\partial\vec{r}}{\partial u_i}$$

$$= \frac{1}{\|\frac{\partial\vec{r}}{\partial u_i}\|}\frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial u_i} = \frac{1}{\|\frac{\partial\vec{r}}{\partial u_i}\|}\frac{\partial\phi}{\partial u_i}$$

$\Rightarrow$ calculate $\vec{\nabla}$:

$$(\vec{\nabla}\phi)_u: \quad (\vec{\nabla}\phi)_u = \frac{1}{\|\frac{\partial\vec{r}}{\partial u}\|}\frac{\partial}{\partial u}\phi = \underline{\underline{\frac{1}{[u^2+v^2]^{\frac{1}{2}}}\frac{\partial}{\partial u}\phi}}$$

$$(\vec{\nabla}\phi)_v: \quad (\vec{\nabla}\phi)_v = \frac{1}{\|\frac{\partial\vec{r}}{\partial v}\|}\frac{\partial}{\partial v}\phi = \underline{\underline{\frac{1}{[u^2+v^2]^{\frac{1}{2}}}\frac{\partial}{\partial v}\phi}}$$

$$(\nabla\phi)_\varphi: \quad (\vec{\nabla}\phi)_\varphi = \frac{1}{\|\frac{\partial\vec{r}}{\partial\varphi}\|}\frac{\partial}{\partial\varphi}\phi = \underline{\underline{\frac{\partial}{\partial\varphi}\phi}}$$

$$\Rightarrow \underline{\underline{\vec{\nabla}\phi(u,v,\varphi) = \left[\frac{1}{[u^2+v^2]^{\frac{1}{2}}}\hat{e}_1\frac{\partial}{\partial u} + \frac{1}{[u^2+v^2]^{\frac{1}{2}}}\hat{e}_2\frac{\partial}{\partial v} + \hat{e}_3\frac{\partial}{\partial\varphi}\right]\phi(u,v,\varphi)}}$$

$$\phi(u,v,\varphi) = u^2 + v^2 - uv$$

$$\vec{\nabla}\phi = \frac{1}{[u^2+v^2]^{\frac{1}{2}}}\hat{e}_1(2u-v) + \frac{1}{[u^2+v^2]^{\frac{1}{2}}}\hat{e}_2(2v-u) + 0$$

$$= \underline{\underline{\frac{1}{[u^2+v^2]^{\frac{1}{2}}}\begin{pmatrix} 2u-v \\ 2v-u \\ 0 \end{pmatrix}}} \quad \text{in this basis.}$$

Aufgabe 4

$$\vec{a} = \begin{pmatrix} x_3 \\ -2x_1 \\ x_2 \end{pmatrix}, \quad \underline{\text{cartesian}} \qquad \underline{\text{vector in cyl. coords = ?}}$$

<u>cyl. coords:</u>

$$\left.\begin{aligned} x_1 &= r\cos\varphi \\ x_2 &= r\sin\varphi \\ x_3 &= z \end{aligned}\right\} \Rightarrow \vec{a} = \begin{pmatrix} z \\ -2r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

<u>basis vectors</u>

$$\frac{1}{\|\frac{\partial\vec{r}}{\partial r}\|}\frac{\partial\vec{r}}{\partial r} = \underline{\underline{\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}}}, \quad \frac{1}{\|\frac{\partial\vec{r}}{\partial\varphi}\|}\frac{\partial\vec{r}}{\partial\varphi} = \frac{1}{r}\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}}$$

$$\frac{1}{\|\frac{\partial\vec{r}}{\partial z}\|}\frac{\partial\vec{r}}{\partial z} = \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}}$$

<u>project onto basis vectors:</u>

$$\left.\begin{aligned} (\vec{a})_r &= \vec{a}\cdot\hat{e}_r = z\cos\varphi - 2r\cos\varphi\sin\varphi \\ (\vec{a})_\varphi &= \vec{a}\cdot\hat{e}_\varphi = -z\sin\varphi - 2r\cos^2\varphi \\ (\vec{a})_z &= \vec{a}\cdot\hat{e}_z = r\sin\varphi \end{aligned}\right\} \Rightarrow \vec{a}_{cyl} = \underline{\underline{\begin{pmatrix} z\cos\varphi - 2r\cos\varphi\sin\varphi \\ -z\sin\varphi - 2r\cos^2\varphi \\ r\sin\varphi \end{pmatrix}}}$$
