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Vektoranalysis (für PhysikerInnen)

SS 2014

4. Übungsblatt

1. April

Aufgabe 1
Gegeben sei die Fläche F in Parameterform

$$\vec{x}(u,v) = \begin{pmatrix} u - \frac{u^3}{3} + uv^2 \\ v - \frac{v^3}{3} + vu^2 \\ u^2 - v^2 \end{pmatrix}, \quad -1 \leq u \leq 1, -1 \leq v \leq 1.$$

Bestimmen Sie den Oberflächeninhalt von F.

Aufgabe 2
Skizzieren Sie die Menge

$$A := \left\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, 0 < xy < 3, x < y < 2x\right\}.$$

Berechnen Sie mittels der Transformation $x(s,t) = \sqrt{\frac{s}{t}}, y(s,t) = \sqrt{st}$ den Flächeninhalt von A und das Integral $\iint_A y^2 dxdy$.

Aufgabe 3
Berechnen Sie unter Verwendung von Polarkoordinaten das Volumen des Bereichs im $\mathbb{R}^3$, der unterhalb des Kegels $z = \sqrt{x^2 + y^2}$ und über der Kreisscheibe $x^2 + y^2 \leq 4$ liegt.

Aufgabe 4
Es sei B der im ersten Oktanten[1] des $\mathbb{R}^3$ liegende Teil des Paraboloids $z = x^2 + y^2$, der nach oben durch die Ebene $z = 4$ begrenzt wird. Bestimmen Sie

$$\iiint_B x dxdydz.$$

Aufgabe 5
Es sei B der Schnitt der Kugel $x^2 + y^2 + z^2 \leq 4$ mit dem Zylinder $x^2 + y^2 \leq 1$. Berechnen Sie

$$\iiint_B x dxdydz.$$

Aufgabe 6
Es seien $p \in \mathbb{R}^3$ und A ein Bereich in der xy-Ebene mit endlichem Flächeninhalt $F(A)$. Beweisen Sie mit Hilfe der Transformationsformel, dass das Volumen des Kegels K über der Grundfläche A mit Spitze in p gegeben ist durch

$$V(K) = \frac{h}{3} F(A),$$

wobei h die Höhe des Kegels ist, d.h. der Abstand von p zur xy-Ebene.

[1] D.h. $x \geq 0, y \geq 0, z \geq 0$.

A. Windisch

Aufgabe 1

$$\vec{x} = \begin{pmatrix} u - \frac{u^3}{3} + uv^2 \\ v - \frac{v^3}{3} + vu^2 \\ u^2 - v^2 \end{pmatrix}, \quad -1 \le u \le 1, \quad -1 \le v \le 1$$

calculate tangent vectors:

$$\frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} 1 - u^2 + v^2 \\ 2vu \\ 2u \end{pmatrix} \qquad \frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} 2uv \\ 1 - v^2 + u^2 \\ -2v \end{pmatrix}$$

determine cross product of tangents to get vectorial area segments:

$$d\vec{\sigma} = \frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\, du\, dv$$

$$\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 1 - u^2 + v^2 & 2vu & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \end{vmatrix} = \begin{pmatrix} -4v^2u - 2u + 2uv^2 - 2u^3 \\ 4u^2v + 2v - 2vu^2 + 2v^3 \\ (1-u^2+v^2)(1-v^2+u^2) - 4v^2u^2 \end{pmatrix}$$

$$= \begin{pmatrix} -2uv^2 - 2u - 2u^3 \\ 2u^2v + 2v + 2v^3 \\ 1 - \cancel{u^2} + \cancel{v^2} - \cancel{v^2} + u^2v^2 - v^4 + \cancel{u^2} - u^4 + u^2v^2 - 4v^2u^2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -2u^3 - 2u(1+v^2) \\ 2v^3 + 2v(1+u^2) \\ 1 - u^4 - v^4 - 2u^2v^2 \end{pmatrix}}}$$

The scalar surface element is obtained by taking the norm of the vectorial surface element.

$$\left\|\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\right\| = \left[(2u^3 + 2u(1+v^2))^2 + (2v^3 + 2v(1+u^2))^2 + (1-u^4-v^4-2u^2v^2)^2\right]^{\frac{1}{2}}$$ ②

$$= \left[(2u(u^2+v^2+1))^2 + (2v(u^2+v^2+1))^2 + (1-(u^2+v^2)^2)^2\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2(4(u^2+v^2)) + (1-(u^2+v^2)^2)^2\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2(4(u^2+v^2)) + ((1-(u^2+v^2))(1+(u^2+v^2)))^2\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2(4(u^2+v^2) + (1-(u^2+v^2))^2\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2(4(u^2+v^2) + (u^2+v^2)^2 - 2(u^2+v^2) + 1)\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2((u^2+v^2)^2 + 2(u^2+v^2)+1)\right]^{\frac{1}{2}}$$

$$= \left[(u^2+v^2+1)^2(u^2+v^2+1)^2\right]^{\frac{1}{2}} = \underline{\underline{(u^2+v^2+1)^2}}$$

The integral becomes

$$\int_{-1}^{1} du \int_{-1}^{1} dv \left\|\frac{\partial \vec{x}}{\partial u} \times \frac{\partial \vec{x}}{\partial v}\right\| = \int_{-1}^{1} du \int_{-1}^{1} dv\, (u^2+v^2+1)^2$$

$$= \int_{-1}^{1} du \int_{-1}^{1} dv \left[(u^2+v^2)^2 + 2u^2 + 2v^2 + 1\right]$$

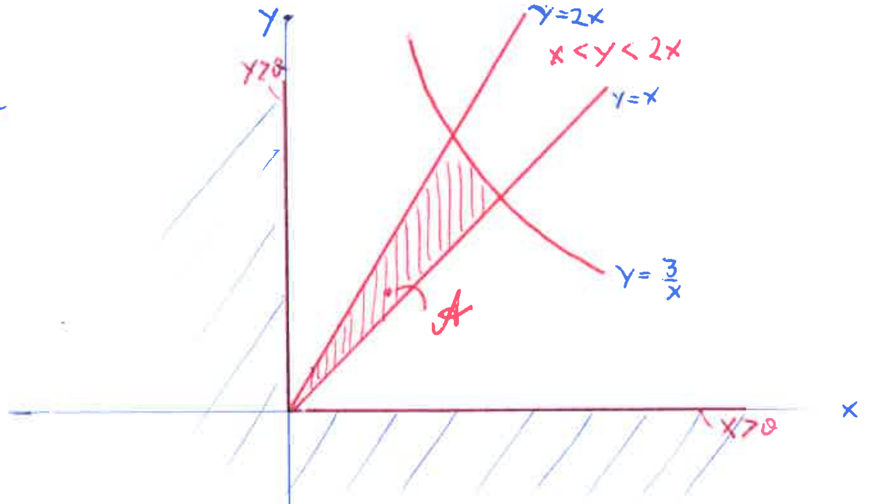
$$= \int_{-1}^{1} du \int_{-1}^{1} dv \left[u^4 + v^4 + 2u^2v^2 + 2u^2 + 2v^2 + 1\right]$$

$$= \int_{-1}^{1} du \left[2u^4 + \frac{2}{5} + 2u^2\frac{2}{3} + 4u^2 + \frac{4}{3} + 2\right]$$

$$= \frac{4}{5} + \frac{4}{5} + \frac{8}{9} + \frac{8}{3} + \frac{8}{3} + 4 = \frac{36}{45} + \frac{36}{45} + \frac{40}{45} + \frac{120}{45} + \frac{120}{45} + \frac{180}{45}$$

$$= \frac{72+180+280}{45} = \underline{\underline{\frac{532}{45}}}$$

Aufgabe 2



$A := \{(x,y) \in \mathbb{R}^2 ;\ x>0,\ y>0,\ 0<xy<3,\ x<y<2x\}$

reparametrize

$$\varphi : \Omega \to A$$

$$\varphi(s,t) = \begin{pmatrix} \sqrt{\frac{s}{t}} \\ \sqrt{st} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \underline{\underline{\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} xy \\ \frac{y}{x} \end{pmatrix}}}$$

$\Rightarrow \quad (x,y) \in A \iff (s,t) \in \Omega,$

Ω:

$$\left.\begin{array}{lll} 0 < xy < 3 & \Rightarrow & \underline{\underline{0<s<3}} \\ x < y < 2x & & \\ 1 < \frac{y}{x} < 2 & \Rightarrow & \underline{\underline{1<t<2}} \end{array}\right\} \Rightarrow \underline{\underline{\Omega = (0,3)\times(1,2)}}$$

surface area:

$$\int_A dx\,dy = \int_\Omega |\det \varphi'(s,t)|\, ds\, dt$$

$$|\det \varphi'(s,t)| = \left| \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \right| = \left| \begin{vmatrix} \frac{1}{2}\frac{1}{\sqrt{st}} & -\frac{1}{2}\frac{\sqrt{s}}{t^{3/2}} \\ \frac{1}{2}\sqrt{\frac{t}{s}} & \frac{1}{2}\sqrt{\frac{s}{t}} \end{vmatrix} \right|$$

$$= \frac{1}{4}\left| \frac{1}{t} + \frac{1}{t} \right| = \underline{\underline{\frac{1}{2t}}}$$

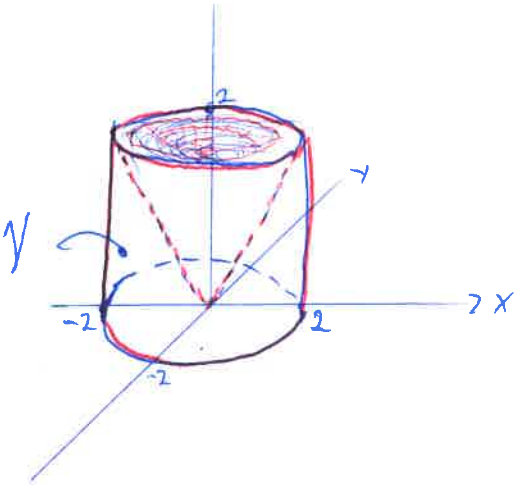
$$\Rightarrow \frac{1}{2}\int_\Omega \frac{1}{t}\, ds\, dt = \frac{1}{2}\int_0^3 ds \int_1^2 dt\, \frac{1}{t} = \frac{3}{2} \log t \Big|_1^2 = \underline{\underline{\frac{3}{2}\log 2}}$$

④

weighted surface area:

$$\int_A y^2 dx\,dy = \int_\Omega s t \frac{1}{2t}\, ds\, dt = \frac{1}{2}\int_0^3 ds \int_1^2 dt\, s = \frac{1}{2}\frac{s^2}{2}\Big|_0^3 = \underline{\underline{\frac{9}{4}}}$$

Aufgabe 3



$x^2+y^2 \leq 4 \quad \Rightarrow \quad \underline{\underline{r=2}}$

$z=\sqrt{x^2+y^2}$

$z_{max} = \sqrt{4} = \underline{\underline{2}}$

trivial calculation:

$$V = V_{cyl} - V_{cone} = r^2\pi z_{max} - \frac{1}{3}r^2\pi z_{max} = \frac{2}{3}r^2\pi z_{max}$$

$$= \frac{2}{3}4\pi \cdot 2 = \underline{\underline{\frac{16\pi}{3}}}$$

calculation using polar coordinates:

$$\vec{r} = r\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ z\end{pmatrix}$$

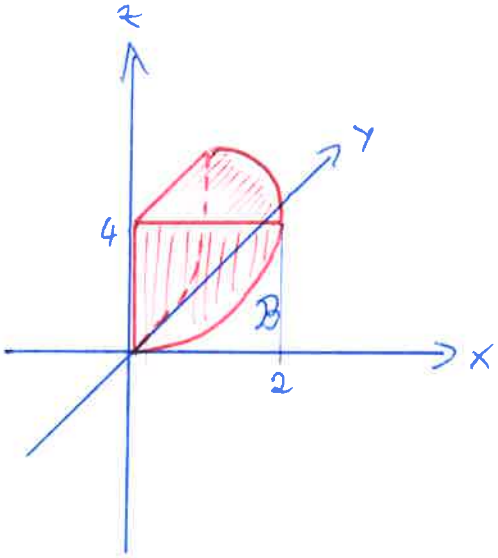
parametrize r boundaries in terms of z:

$z \leq r \leq 2$ z_{max}

$$V = \int_V dV = \int_0^2 dz \int_0^{2\pi} d\varphi \int_z^2 dr\, r = 2\pi\int_0^2 dz\, \frac{1}{2}\left[4-z^2\right] = 4\pi\cdot 2 - \frac{\pi}{3}z^3\Big|_0^2$$

$$= 8\pi - \frac{8\pi}{3} = \frac{24\pi}{3} - \frac{8\pi}{3} = \underline{\underline{\frac{16\pi}{3}}}$$

Aufgabe 4



use polar coordinates:

$\varphi \in [0, \frac{\pi}{2}]$

$z \in [0, 4]$

$r \in [0, 2]$

Reparametrize r in terms of z:

$z = x^2 + y^2 = r^2 \Rightarrow \underline{\underline{r = \sqrt{z}}}$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

check: use formula for volume of paraboloid:

$\frac{1}{4} V_{par} = \frac{1}{4} \cdot \frac{r_{max}^2 \pi}{2} h = \frac{1}{4} \frac{r_{max}^2 \pi}{2} 4 \quad \frac{4\pi 4}{8} = \underline{\underline{2\pi}}$

volume via polar coordinates:

$$\int_0^4 \int_0^{\pi/2} \int_0^{\sqrt{z}} r\,dr\,d\varphi\,dz = \frac{\pi}{2} \frac{1}{2} \int_0^4 dz\, z = \frac{\pi}{4} \frac{16}{2} = \underline{\underline{2\pi}}$$

Weighted integral over B:

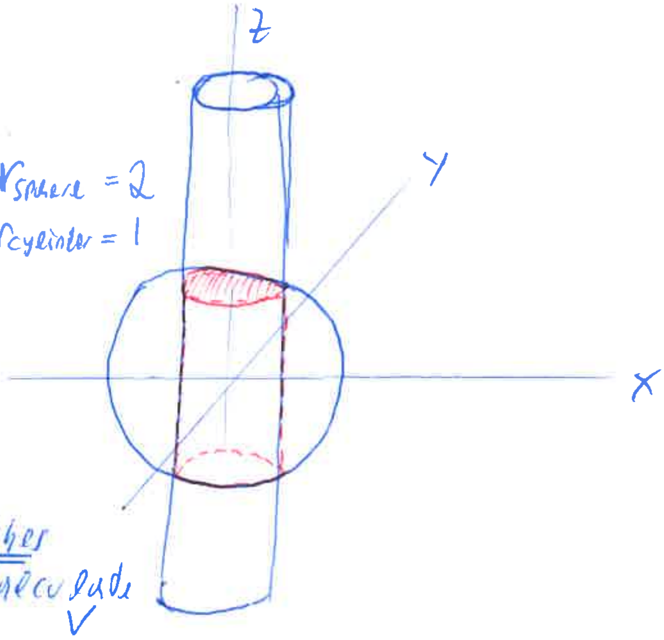
$$\int_0^4 \int_0^{\pi/2} \int_0^{\sqrt{z}} r^2 \cos\varphi\,dr\,d\varphi\,dz = \int_0^4 dz \underbrace{\int_0^{\pi/2} d\varphi \cos\varphi}_{=1} \frac{z^{3/2}}{3} = \frac{2}{5} \frac{z^{5/2}}{3} \Big|_0^4$$

$$= \frac{2}{15} 32 = \underline{\underline{\frac{64}{15}}} \quad \sim 4{,}266$$

Aufgabe 5

$\mathcal{B}: \quad x^2+y^2+z^2 \leq 4 \quad \Rightarrow r_{sphere} = 2$

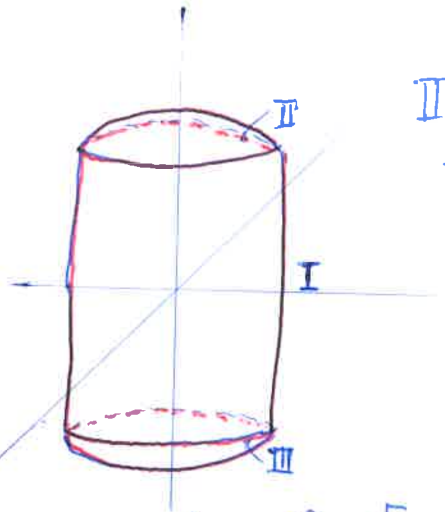
$\qquad x^2+y^2 \leq 1 \qquad\qquad r_{cylinder} = 1$



$$\iiint_{\mathcal{B}} x \, dx\, dy\, dz = 0$$

because one integrates an antisymmetric function over a symmetric interval → x-int vanishes → overall integral is zero ⇒ calculate ✓

Split into 3 subsets:



II & III: use spherical coords.

I: use cylindrical coords.

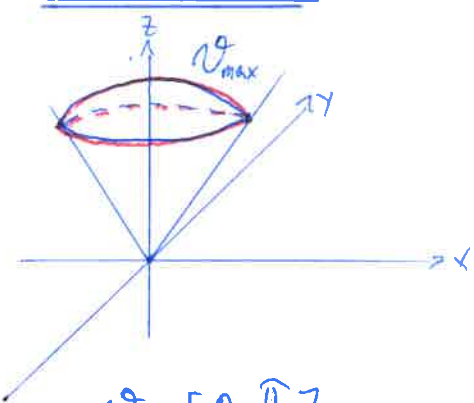
Region I:

$$\left.\begin{array}{l} x^2+y^2+z^2=4 \\ x^2+y^2=1 \end{array}\right\} \Rightarrow z = \pm\sqrt{3}$$

$$\text{I}: \iiint_{\text{I}} x\,dx\,dy\,dz = \int_0^1 dr\, r \int_0^{2\pi} d\varphi \int_{-\sqrt{3}}^{\sqrt{3}} dz\, \underbrace{r\cos\varphi}_{=x} = 0 \qquad V_{\text{I}} = \pi r^2 h = \pi 2\sqrt{3}$$

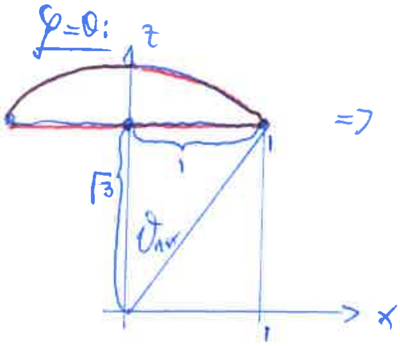
Region II:

$\varphi \in [0, 2\pi)$



$\Rightarrow \vartheta \in [0, \frac{\pi}{6}]$

$\varphi = 0$:



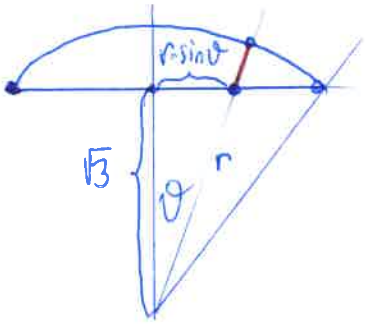
$$\Rightarrow \vartheta_{max} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

(7)

Boundaries for r:

upper boundary: radius of sphere $\Rightarrow$ $\underline{\underline{r_{max} = 2}}$

lower boundary:



$\Rightarrow r^2 = r^2\sin^2\vartheta + 3$

$\Rightarrow r = \sqrt{\frac{3}{1-\sin^2\vartheta}} = \sqrt{\frac{3}{\cos^2\vartheta}}, \quad \vartheta \in [0, \frac{\pi}{6}]$

Parametrization II:

$r \in [\sqrt{\frac{3}{\cos^2\vartheta}}, 2]$ $\qquad x = r\sin\vartheta\cos\varphi$

$\vartheta \in [0, \frac{\pi}{6}]$ $\qquad y = r\sin\vartheta\sin\varphi$

$\varphi \in [0, 2\pi)$ $\qquad z = r\cos\vartheta$

$dx\,dy\,dz = r^2\sin\vartheta\, dr\, d\vartheta\, d\varphi$

$$\Rightarrow \iiint_{II} x\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} d\vartheta \int_{\sqrt{\frac{3}{\cos^2\vartheta}}}^{2} dr \underbrace{r^2\sin\vartheta}_{\det J} \underbrace{r\sin\vartheta\cos\varphi}_{=x}$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int_{\sqrt{\frac{3}{\cos^2\vartheta}}}^{2} dr\, r^3 \sin^2\vartheta\cos\varphi = \underline{\underline{0}}$$ (due to cosine)

$$V_{II} = \int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int_{\sqrt{\frac{3}{\cos^2\vartheta}}}^{2} dr\, r^2\sin\vartheta = 2\pi \int_0^{\pi/6} d\vartheta \sin\vartheta \left.\frac{r^3}{3}\right|_{\sqrt{\frac{3}{\cos^2\vartheta}}}^{2}$$

$$= \frac{2\pi}{3}\int_0^{\pi/6} d\vartheta \sin\vartheta \left(8 - \left(\frac{3}{\cos^2\vartheta}\right)^{3/2}\right) = \frac{16\pi}{3}\left(-\cos\vartheta\right)\Big|_0^{\frac{\pi}{6}} - \frac{2\pi}{3}(3)^{3/2}\int_0^{\frac{\pi}{6}} d\vartheta \frac{\sin\vartheta}{\cos^3\vartheta}$$

$$= -\frac{16\pi}{3}\frac{\sqrt{3}}{2} + \frac{16\pi}{3} - \underbrace{\frac{2\pi}{\cancel{3}}\cancel{(3)}^{3/2}}_{2\pi\sqrt{3}} \underbrace{\int_0^{\frac{\pi}{6}} d\vartheta \frac{\sin\vartheta}{\cos^3\vartheta}}_{\tilde{I}}$$

$$\tilde{I} = \int_0^{\frac{\pi}{6}} d\vartheta \, \frac{\sin\vartheta}{\cos^3\vartheta} = \left| \begin{array}{l} u = \cos\vartheta \\ du = -\sin\vartheta \, d\vartheta \\ 0 \to 1 \quad \frac{\pi}{6} \to \frac{\sqrt{3}}{2} \end{array} \right| = -\int_1^{\frac{\sqrt{3}}{2}} du \, \frac{1}{u^3}$$

$$= \frac{1}{2} u^{-2} \Big|_1^{\sqrt{3}/2} = \frac{1}{2}\left(\frac{4}{3} - 1\right) = \underline{\underline{\frac{1}{6}}}$$

$$\Rightarrow V_{II} = \frac{16\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right) - \frac{2\pi}{3} 3^{3/2} \frac{1}{6}$$

$$= \frac{16\pi}{3} - \frac{\overset{8}{\cancel{16}}\pi\sqrt{3}}{\cancel{6}\,3} - \frac{\cancel{2}\pi\sqrt{3}}{\underset{3}{\cancel{6}}}$$

$$= \frac{16\pi}{3} - \frac{\overset{3}{\cancel{9}}\pi\sqrt{3}}{\cancel{3}} = \underline{\underline{2\pi\left(\frac{8}{3} - \frac{3\sqrt{3}}{2}\right)}}$$

The overall volume is then

$$V_B = V_I + V_{II} + V_{III} = V_I + 2V_{II} = 2\pi\sqrt{3} + 2\pi\left(\frac{8}{3} - \frac{3\sqrt{3}}{2}\right)$$

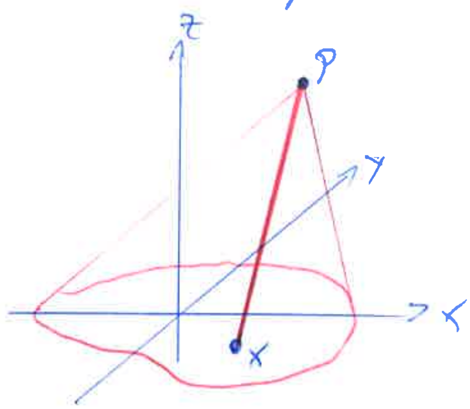
$$= 2\pi\sqrt{3} + \frac{16\pi}{3} - 3\pi\sqrt{3} = \underline{\underline{\frac{16\pi}{3} - \sqrt{3}\pi}} \approx 11.3$$

Aufgabe 6

$P \in \mathbb{R}^3$, $A \in \mathbb{R}^2$ (x-y-plane)

$F(A) > 0$, area.

Show that $V(K) = \frac{h}{3} F(A)$, K ... cone over A with apex P



w.l.o.g.: Let $h > 0$.

Introduce mapping

$$\phi : A \times [0, h] \to K(A, P)$$

$$(x, z) \mapsto \underbrace{\left(x + \frac{z}{h}(P - x), z\right)}_{\substack{\text{line connecting} \\ x \text{ and } P,\ \text{see sketch}}}$$

ϕ is a bijective continuous differentiable function. ⑨

$$\Rightarrow |\det \phi'(x,z)| = \left| \begin{vmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial z} \end{vmatrix} \right|$$

unit matrix 2×2

z-comp

$$= \left| \begin{vmatrix} \mathbb{1}(1-\frac{z}{h}) & (p-x)_z \frac{1}{h} \\ 0 & 1 \end{vmatrix} \right|$$

$$= \underline{\underline{\left(1-\frac{z}{h}\right)^2}}$$

$$V(K(A,p)) = \int_{K(A,p)} dV = \int_{A\times[0,h]} \left(1-\frac{z}{h}\right)^2 dx\, dz$$

$$= \int_0^h dz \int_A \left(1-\frac{z}{h}\right)^2 dx = \underbrace{\int_A dx}_{\mathcal{F}(A)} \int_0^h dz \left(1-\frac{z}{h}\right)^2$$

$$= \mathcal{F}(A) \int_0^h dz \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) = \mathcal{F}(A)\left[h - h + \frac{h}{3}\right]$$

$$= \underline{\underline{\frac{h}{3}\mathcal{F}(A)}}$$