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Vektoranalysis (für PhysikerInnen)

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3. Übungsblatt

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Aufgabe 1
Berechnen Sie das Doppelintegral

$$\iint_B \Big(xy + \frac{1}{(1+x+y)^2}\Big)dxdy,$$

wobei B das Innere des Dreiecks mit den Eckpunkten $(0,0),(0,1)$ und $(2,2)$ bezeichnet.

Aufgabe 2
Bestimmen Sie den Wert des Integrals

$$\iint_B (x^2+y^2)dxdy,$$

wobei B die zwischen den Parabeln $y = x^2$ und $y = 3x^2 - 2$ eingeschlossene Teilmenge des $\mathbb{R}^2$ ist; skizzieren Sie den Bereich B.

Aufgabe 3
Skizzieren Sie die Menge

$$S := \Big\{(x,y) : -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}, \frac{1}{2} \leq y \leq \cos x\Big\}$$

und bestimmen Sie das Flächenträgheitsmoment

$$I_y := \iint_S x^2 dxdy$$

mit der y-Achse als Drehachse und x dem Abstand von der Drehachse.

Aufgabe 4
Es sei K der Körper im $\mathbb{R}^3$, der zwischen der xy-Ebene und der Ebene $x + 2y + z = 10$ im Inneren des (unendlich langen) Zylinders $x^2 + y^2 = 4$ liegt. Skizzieren Sie K und berechnen Sie das Volumen von K, d.h. berechnen Sie

$$\mathrm{Vol}(K) = \iiint_K 1dxdydz.$$

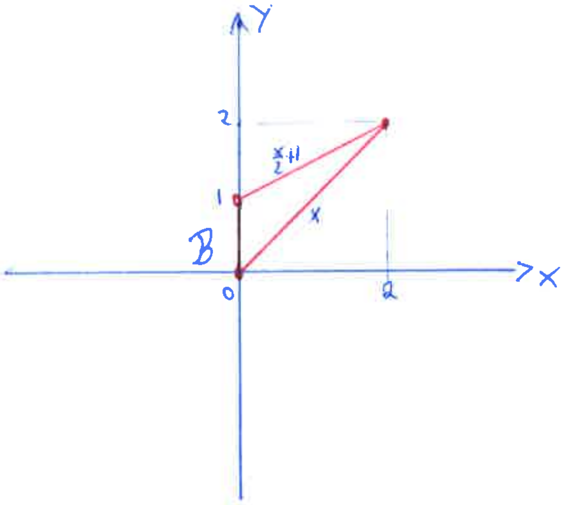
Aufgabe 5
Skizzieren Sie den Bereich B, der vom Paraboloid $x = 4y^2 + 4z^2$ und der Ebene $x = 4$ begrenzt wird, und berechnen Sie

$$\iiint_B xdxdydz.$$

A. Windisch

Aufgabe 1

triangle, $(0,0), (0,1), (2,2)$



$$\iint_B \left(xy + \frac{1}{(1+x+y)^2}\right) dx\, dy$$

$$= \int_0^2 dx \int_x^{\frac{x}{2}+1} dy \left[xy + \frac{1}{(1+x+y)^2}\right]$$

$$\Rightarrow \int_0^2 dx \int_x^{\frac{x}{2}+1} dy\, xy = \int_0^2 dx\, x \left[\frac{y^2}{2}\right]\Bigg|_x^{\frac{x}{2}+1} = \int_0^2 dx \left[\frac{\left(\frac{x}{2}+1\right)^2}{2} - \frac{x^2}{2}\right] x$$

$$= \frac{1}{2}\int_0^2 dx \left[\frac{x^2}{4} + x + 1 - x^2\right] x = \frac{1}{2}\int_0^2 dx\, x \left[-\frac{3}{4}x^2 + x + 1\right] = \frac{1}{2}\int_0^2 dx \left(-\frac{3}{4}x^3 + x^2 + x\right)$$

$$= \frac{1}{2}\left(-\frac{3}{4}\frac{x^4}{4}\right)\Bigg|_0^2 + \frac{1}{2}\frac{x^3}{3}\Bigg|_0^2 + \frac{x^2}{4}\Bigg|_0^2 = -\frac{3}{32}\cancel{16}_2 + \frac{4}{3} + 1 = -\frac{2}{6} + \frac{8}{6} + \frac{6}{6} = \underline{\underline{\frac{5}{6}}}$$

$$\int_0^2 dx \int_x^{\frac{x}{2}+1} dy \frac{1}{(x+y+1)^2} = \left| \begin{array}{l} u = x+y+1 \\ du = dy \\ x \to 2x+1 \\ \frac{x}{2}+1 \to \frac{3x}{2}+2 \end{array} \right| = \int_0^2 dx \int_{2x+1}^{\frac{3x}{2}+2} du \frac{1}{u^2} = \int_0^2 dx \left(-\frac{1}{u}\right)\Bigg|_{2x+1}^{\frac{3x}{2}+2}$$

$$= \int_0^2 dx \frac{-1}{\frac{3}{2}x+2} + \int_0^2 dx \frac{1}{2x+1}$$

$$\Rightarrow \int_0^2 dx \frac{-2}{3x+4} = \left| \begin{array}{l} v = 3x+4 \\ dv = 3dx \Rightarrow dx = \frac{dv}{3} \\ 0 \to 4 \\ 2 \to 10 \end{array} \right| = \int_4^{10} dv \left(-\frac{2}{3}\right)\frac{1}{v} = -\frac{2}{3}\log v\Bigg|_4^{10}$$

$$= -\frac{2}{3}\log\frac{10}{4} = \underline{\underline{-\frac{2}{3}\log\frac{5}{2}}}$$

$$\Rightarrow \int_0^2 dx \frac{1}{2x+1} = \left| \begin{array}{l} w = 2x+1 \\ dw = 2dx \Rightarrow dx = \frac{dw}{2} \\ 0 \to 1, \quad 2 \to 5 \end{array} \right| = \frac{1}{2}\int_1^5 dw \frac{1}{w} = \underline{\underline{\frac{1}{2}\log 5}}$$

(2)

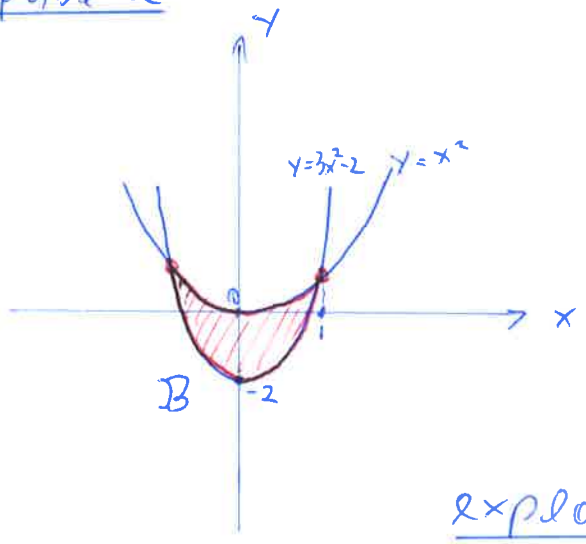
$$\Rightarrow \int_0^2 dx \int_x^{\frac{x}{2}+1} dy \left[xy + \frac{1}{(x+y+1)^2} \right] = -\frac{2}{3}\log\frac{5}{2} + \frac{1}{2}\log 5 + \frac{5}{6}$$

$$= \frac{5}{6} - \frac{2}{3}\log 5 + \frac{2}{3}\log 2 + \frac{1}{2}\log 5 = \frac{5}{6} - \frac{1}{6}\log 5 + \frac{2}{3}\log 2$$

$$= -\frac{1}{6}\log 5 + \frac{1}{3}\log 4 + \frac{5}{6} = \frac{5}{6} - \frac{1}{6}\log 5 + \frac{1}{6}\log 16$$

$$= \frac{5}{6} + \frac{1}{6}\log\frac{16}{5} = \underline{\underline{\frac{1}{6}\left(5 + \log\frac{16}{5}\right)}}$$

Aufgabe 2



intersection

$x^2 = 3x^2 - 2$

$-2x^2 = -2$

$x^2 = 1 \Rightarrow \underline{\underline{x = \pm 1}}$

exploit symmetry

$$2\int_0^1 dx \int_{3x^2-2}^{x^2} dy\,(x^2+y^2) = 2\int_0^1 dx \left[x^2(x^2-3x^2+2) + \frac{y^3}{3}\Big|_{3x^2-2}^{x^2} \right]$$

$$= 2\int_0^1 dx \left[x^4 - 3x^4 + 2x^2 + \frac{x^6}{3} - \frac{1}{3}(3x^2-2)^3 \right]$$

$$= 2\int_0^1 dx \left[-2x^4 + 2x^2 + \frac{x^6}{3} + \frac{8}{3} + 18x^4 - 12x^2 - 9x^6 \right]$$

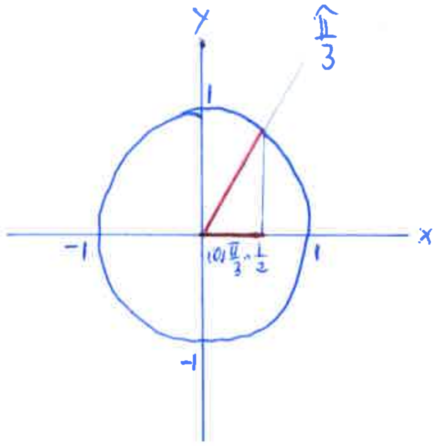
$$= 2\int_0^1 dx \left[-\frac{26}{3}x^6 + 18x^4 - 2x^4 + 2x^2 - 12x^2 + \frac{8}{3} \right] = 2\int_0^1 dx \left[-\frac{26}{3}x^6 + 16x^4 - 10x^2 + \frac{8}{3} \right]$$

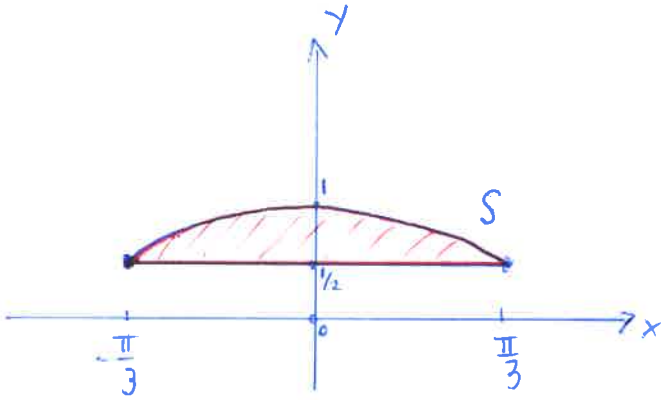
$$= -\frac{52}{21} + \frac{32}{5} - \frac{4}{3} = -\frac{260}{105} + \frac{672}{105} - \frac{140}{105} = \underline{\underline{\frac{272}{105}}}$$

Aufgabe 3

$$S := \{(x,y) : -\tfrac{\pi}{3} \leq x \leq \tfrac{\pi}{3},\ \tfrac{1}{2} \leq y \leq \cos x\}$$

$$\cos\frac{\pi}{3} = \frac{1}{2}$$





exploit symmetry

second moment of area:

$$I_y := 2\int_0^{\frac{\pi}{3}} dx \int_{\frac{1}{2}}^{\cos x} dy\, x^2 = 2\int_0^{\frac{\pi}{3}} dx\, x^2 (\cos x - \tfrac{1}{2}) = 2\int_0^{\frac{\pi}{3}} dx\, x^2 \cos x - \frac{\pi^3}{27}\frac{1}{3} =$$

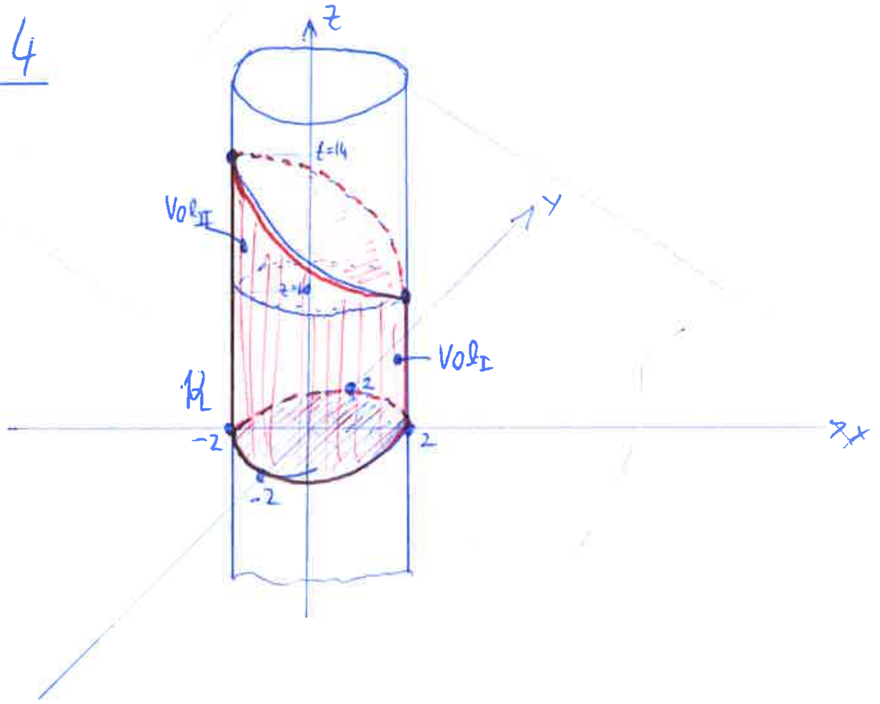
$$= 2\int_0^{\frac{\pi}{3}} dx\, x^2 \cos x - \frac{\pi^3}{81}$$

$$2\int_0^{\frac{\pi}{3}} dx\, x^2 \cos x = 2\sin x\, x^2 \Big|_0^{\frac{\pi}{3}} - 4\int_0^{\frac{\pi}{3}} dx\, x \sin x = \frac{\sqrt{3}}{9}\pi^2 - 4\left[-x\cos x \Big|_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} dx \cos x\right]$$

$$= \frac{\sqrt{3}\pi^2}{9} - 4\left[-\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}\pi^2}{9} + \frac{2\pi}{3} - \frac{4\sqrt{3}}{2} = \underline{\underline{\frac{\pi^2}{3\sqrt{3}} + \frac{2\pi}{3} - 2\sqrt{3}}}$$

$$\Rightarrow \quad I_y = 2\int_0^{\frac{\pi}{3}} dx\, x^2 (\cos x - \tfrac{1}{2}) = \underline{\underline{-\frac{\pi^3}{81} + \frac{\pi^2}{3\sqrt{3}} + \frac{2\pi}{3} - 2\sqrt{3}}}$$

Aufgabe 4



Parametrize cylinder: $2\begin{pmatrix}\cos t\\ \sin t\\ z\end{pmatrix}$ $t\in[0,2\pi)$

Parametrize plane: $z = 10 - x - 2y$

$$\Rightarrow \quad z(t) = 10 - 2\cos t - 4\sin t$$

$\Rightarrow$ upper bound of K: $\begin{pmatrix}2\cos t\\ 2\sin t\\ 10-2\cos t-4\sin t\end{pmatrix}$

$z_{min} = 6 \quad (t=\frac{\pi}{2})$

$z_{max} = 14 \quad (t=\frac{3\pi}{2})$

$$Vol_K = Vol_I + Vol_{II} = r^2\pi z_{min} + Vol_{II} = 24\pi + Vol_{II}$$

$$Vol_{II} = \int dx \int dy \int_{Vol_{II}} dz \underset{\substack{\uparrow\\ [\text{Jacobian: } dx\,dy\,dz = r\,dr\,dy\,dz]}}{=} \int_0^2 dr \int_0^{2\pi} dt \int_6^{10-2\cos t-4\sin t} dz\, r = \int_0^2 dr\, r \int_0^{2\pi} dt\,[10-2\cos t-4\sin t-6]$$

$$= 2\int_0^{2\pi}[4-2\cos t-4\sin t] = 2[8\pi] = 16\pi$$

$$Vol_K = Vol_I + Vol_{II} = 24\pi + 16\pi = \underline{\underline{40\pi}}$$

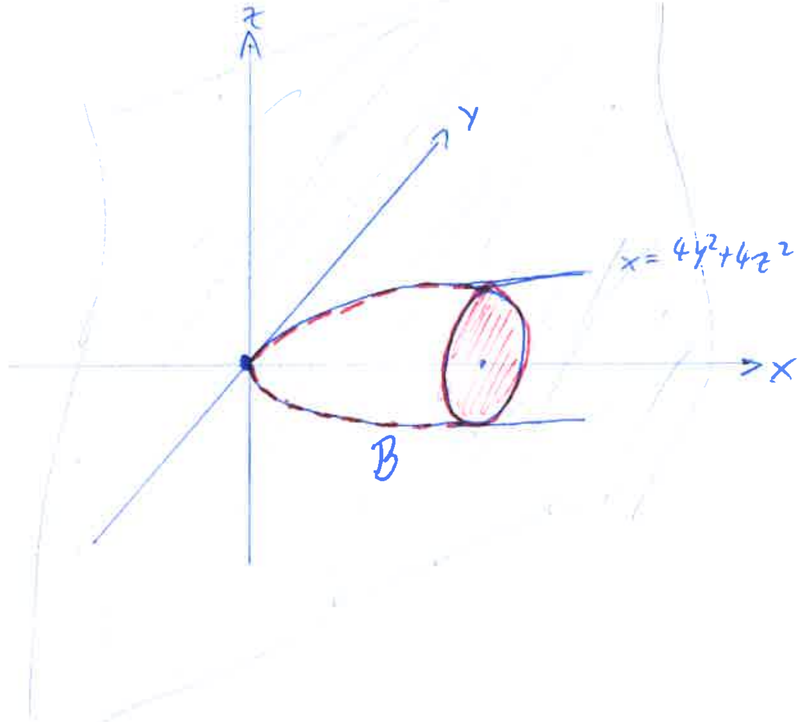
Aufgabe 5

<u>Paraboloid:</u>

$x = 4y^2 + 4z^2$

<u>plane:</u>

$x = 4$



<u>strategy:</u>

switch coordinates such that $x \to z$, $y \to x$, $z \to y$ and integrade B using polar coordinates.

$$\Rightarrow \quad z = 4x^2 + 4y = 4(x^2 + y^2) = 4r^2 \quad \Rightarrow \quad \underline{\underline{r = \sqrt{\tfrac{z}{4}}}}$$

<u>volume element</u>: $dx\,dy\,dz \to r\,dr\,d\varphi\,dz$

$$\Rightarrow \iiint_B dx\,dy\,dz\,x = \int_0^4 dz\,z \int_0^{\sqrt{\frac{z}{4}}} dr\,r \int_0^{2\pi} d\varphi = \frac{2\pi}{2} \int_0^4 dz\,\frac{z^2}{4}$$

$$= \frac{\pi}{4}\,\frac{z^3}{3}\Big|_0^4 = \frac{\pi}{12}\,64 = \underline{\underline{\frac{16\pi}{3}}}$$