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Vektoranalysis (für PhysikerInnen)

SS 2014

2. Übungsblatt

18. März

Aufgabe 1

(i) Berechnen Sie die Bogenlänge der sogenannten Traktrix

$$x = a \operatorname{arcosh}\left(\frac{a}{y}\right) - \sqrt{a^2 - y^2}$$

mit $0 < b \leq y \leq a$ und $a, b \in \mathbb{R}$.

(ii) Berechnen Sie die Bogenlänge der in Polarkoordinaten gegebenen Kurve

$$r = \frac{e^{\varphi} - 1}{e^{\phi} + 1}$$

mit $0 \leq \phi \leq \alpha$ und $\alpha > 0$.

Aufgabe 2

(i) Parametrisieren[1] und skizzieren Sie die Kurve $K = \{(x, y)^T \in \mathbb{R}^2 : x^3 - y + 1 = 0\}$.

(ii) Parametrisieren Sie die Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ mit $a, b > 0$ und berechnen Sie ihre Krümmung. In welchen Punkten nimmt die Krümmung ihr Maximum an?

Aufgabe 3
Bestimmen Sie Parameterform und Tangente folgender Kurven, die in Polarkoordinaten gegeben sind.

(i) $r = \phi$ mit $\phi \geq 0$;

(ii) $r = \ln \phi$ mit $\phi \geq 1$.

Aufgabe 4
Gegeben sei die Kurve

$$\vec{r}(t) = e^{at} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad t \geq 0,$$

wobei $a > 0$.

(i) Skizzieren Sie die Kurve und berechnen Sie die Weglänge von $t = 0$ bis $t = 2\pi$.

(ii) Berechnen Sie das begleitende Dreibein im Punkt $t = 1$.

(iii) Berechnen Sie die Krümmung und die Torsion der Kurve.

Bitte wenden!

[1] D.h. finden Sie eine Parameterdarstellung für diese Kurve.

Aufgabe 5

Ein elektrisch geladenes Teilchen bewegt sich in einem homogenen Magnetfeld auf der Bahn

$$\vec{r}(t) = \begin{pmatrix} a\cos(\omega t) \\ a\sin(\omega t) \\ ct \end{pmatrix}, \quad t \geq 0,$$

wobei $a, \omega, c > 0$.

(i) Skizzieren Sie die Bahnkurve und berechnen Sie die Länge der Strecke, die das Teilchen in der Zeit von $t = 0$ bis $t = \frac{2\pi}{\omega}$ zurücklegt.

(ii) Bestimmen Sie die Krümmung und den Hauptnormalenvektor.

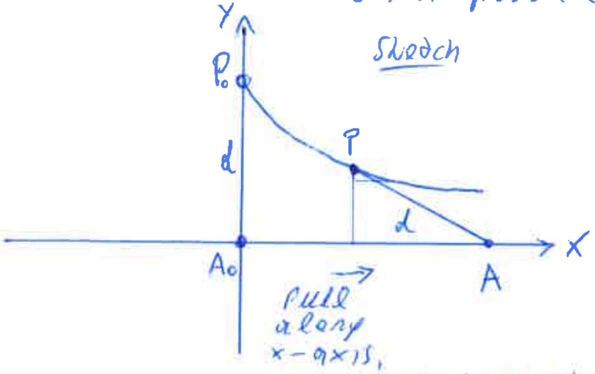
(iii) Berechnen Sie das begleitende Dreibein und die Torsion.

Lösungen zum 2. Übungsblatt aus Vektoranalysis SS14 (1)

A. Windisch

Aufgabe 1

(i) tractrix (movement of object under friction when pulled, see sketch)

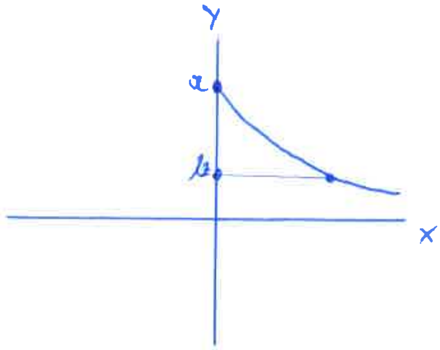


arclength tractrix:

connection to point fixed
$\Rightarrow$ tangent to $\overline{PA} = d = \text{const}$

$$X = a \operatorname{arcosh}\frac{a}{y} - \sqrt{a^2-y^2}$$

$$0 < b \leq y \leq a \ , \quad a, b \in \mathbb{R}$$



continuously differentiable on $[a,b]$

$\Rightarrow$ rectifiable $\Rightarrow$ arclength $l = \int_a^b dy \sqrt{1+(f'(y))^2}$

$$f(y) = a \operatorname{arcosh}\frac{a}{y} - \sqrt{a^2-y^2}$$

$$f'(y) = -\frac{a^2}{\sqrt{\frac{a}{y}-1}\sqrt{\frac{a}{y}+1}\; y^2} + \frac{y}{\sqrt{a^2-y^2}} = \frac{y}{\sqrt{a^2-y^2}} - \left(\frac{a}{y}\right)^2 \frac{1}{\sqrt{\left(\frac{a}{y}\right)^2-1}}$$

$$= \frac{y}{\sqrt{a^2-y^2}} - \frac{a^2}{y}\frac{1}{\sqrt{a^2-y^2}} = \frac{y^2-a^2}{y\sqrt{a^2-y^2}} = -\frac{a^2-y^2}{y\sqrt{a^2-y^2}} = \underline{\underline{-\frac{\sqrt{a^2-y^2}}{y}}}$$

integrand: $\left[1+(f'(y))^2\right]^{\frac{1}{2}} = \left[1+\frac{a^2-y^2}{y^2}\right]^{\frac{1}{2}} = \underline{\underline{\left[\frac{a^2}{y^2}\right]^{\frac{1}{2}}}}$

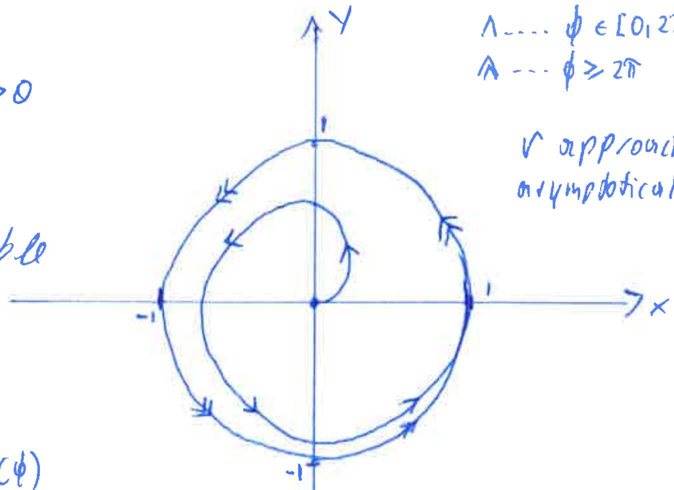
$\Rightarrow$ arclength tractrix:

$$l = \int_b^a dy \sqrt{\frac{a^2}{y^2}} = a\int_b^a dy\, \frac{1}{y} = a \ln a - a \ln b$$

$$= \underline{\underline{a \ln\left(\frac{a}{b}\right)}}$$

(ii)

$$r = \frac{e^{\phi}-1}{e^{\phi}+1} \qquad 0 \le \phi \le \alpha \,, \quad \alpha > 0$$



$\wedge$ .... $\phi \in [0, 2\pi)$
$\wedge$ ... $\phi \geq 2\pi$

2

r approaches 1 asymptotically

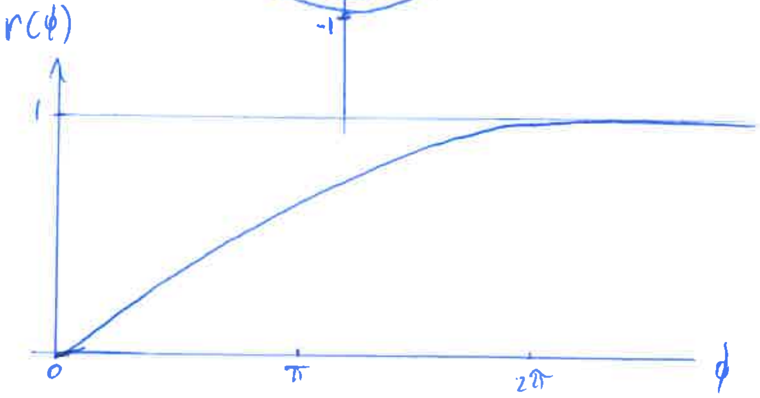
$r(\phi)$ is continuous differentiable

$\Rightarrow$ arclength $l = \int_0^{\alpha} d\phi \sqrt{r^2 + (r')^2}$

$$r'(\phi) = \frac{e^{\phi}(e^{\phi}+1) - e^{\phi}(e^{\phi}-1)}{(e^{\phi}+1)^2}$$

$$= \frac{\cancel{e^{2\phi}} + e^{\phi} - \cancel{e^{2\phi}} + e^{\phi}}{(e^{\phi}+1)^2} = \underline{\underline{\frac{2e^{\phi}}{(e^{\phi}+1)^2}}}$$

$$\left[= \frac{2e^{\phi}}{e^{2\phi}+2e^{\phi}+1} = \frac{2}{e^{\phi}+2+e^{-\phi}} = \frac{2}{2(\cosh(\phi)+1)} = \underline{\underline{\frac{1}{1+\cosh\phi}}} \right]$$



$$r^2 + (r')^2 = \frac{(e^{\phi}-1)^2}{(e^{\phi}+1)^2} + \frac{4e^{2\phi}}{(e^{\phi}+1)^4} = \frac{(e^{\phi}-1)^2(e^{\phi}+1)^2 + 4e^{2\phi}}{(e^{\phi}+1)^4}$$

$$= \frac{e^{4\phi} - \cancel{2e^{3\phi}} + e^{2\phi} + \cancel{2e^{3\phi}} - \cancel{4e^{2\phi}} + \cancel{2e^{\phi}} + e^{2\phi} - \cancel{2e^{\phi}} + 1 + \cancel{4e^{2\phi}}}{(e^{\phi}+1)^4}$$

$$= \frac{e^{4\phi} + 2e^{2\phi} + 1}{(e^{\phi}+1)^4} = \underline{\underline{\frac{(e^{2\phi}+1)^2}{(e^{\phi}+1)^4}}}$$

$$\Rightarrow l = \int_0^{\alpha} d\phi \sqrt{\frac{(e^{2\phi}+1)^2}{(e^{\phi}+1)^4}} = \int_0^{\alpha} d\phi \, \frac{e^{2\phi}+1}{(e^{\phi}+1)^2} = \left| \begin{array}{l} e^{\phi} = \xi \\ d\xi = e^{\phi} d\phi \\ 0 \to 1 \,, \ \alpha \to e^{\alpha} \\ e^{-\phi} = \xi^{-1} \end{array} \right| = \int_1^{e^{\alpha}} d\xi \, \frac{\xi + \xi^{-1}}{(\xi+1)^2}$$

$$= \int_1^{e^{\alpha}} d\xi \, \frac{\xi^2+1}{\xi(\xi+1)^2} \stackrel{?}{=} \int_1^{e^{\alpha}} d\xi \left[\frac{A}{\xi} + \frac{B}{\xi+1} + \frac{C}{(\xi+1)^2} \right]$$

$$A(\xi+1)^2 + B(\xi(\xi+1)) + C\xi \stackrel{?}{=} \xi^2 + 1$$

$$A\xi^2 + 2A\xi + A + B\xi^2 + B + C\xi \stackrel{?}{=} \xi^2 + 1$$

$$\Rightarrow \left. \begin{array}{l} A + B = 1 \\ 2A + C = 0 \end{array} \right\} \Rightarrow A = 1, \ B = 0, \ C = -2$$

③

$$\Rightarrow \int_1^{e^\alpha} d\xi \left[\frac{1}{\xi} - \frac{2}{(\xi+1)^2}\right] = \ln\xi \Big|_1^{e^\alpha} + \frac{2}{\xi+1}\Big|_1^{e^\alpha} = \alpha + \frac{2}{e^\alpha+1} - 1$$

$$= \alpha + \frac{2-e^\alpha-1}{e^\alpha+1} = \alpha + \frac{-e^\alpha+1}{e^\alpha+1} = \underline{\underline{\alpha - r(\alpha)}}$$

Aufgabe 2

(i) $K = \{(x,y)^T \in \mathbb{R}^2 : x^3 - y + 1 = 0\}$

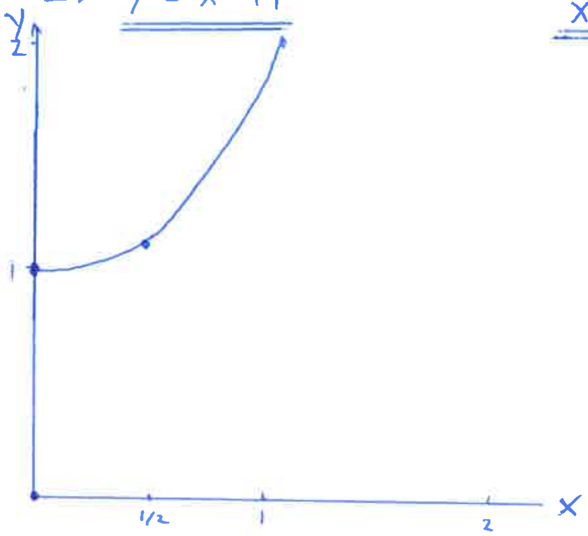
$x^3 - y + 1 = 0$

$\Rightarrow \underline{\underline{y = x^3 + 1}}$

<u>use x as parameter:</u>

$\underline{\underline{x(t) = t\,;\ y(t) = t^3 + 1}}$

$\underline{t=0}$: $x = 0;\ y = 1$
$\underline{t = \frac{1}{2}}$: $x = \frac{1}{2};\ y = 1{,}125$
$\underline{t = 1}$: $x = 1;\ y = 2$



(ii) <u>Ellipse:</u> $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a,b > 0$

introduce parameter t such that: $x = a\cos t \quad y = b\sin t,\ t \in [0, 2\pi)$

$$\Rightarrow \frac{a^2\cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = 1 \quad \checkmark$$

$$\Rightarrow \underline{\underline{x(t) = a\cos(t)\ ;\ y(t) = b\sin(t)}}$$

<u>maximal curvature:</u> $\frac{dk(t)}{dt} \overset{!}{=} 0, \quad \frac{d^2k(t)}{dt^2} < 0$

<u>curvature k(t):</u> $k(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}, \quad \dot{\ } = \frac{d}{dt}, \quad \ddot{\ } = \frac{d^2}{dt^2}$

④

$\dot{x}(t) = -a\sin t$; $\dot{y}(t) = b\cos t$; $\ddot{x}(t) = -a\cos t$; $\ddot{y}(t) = -b\sin(t)$

$$\Rightarrow \kappa(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{(-a\sin t)(-b\sin t) - b\cos t(-a\cos t)}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

$$= \frac{ab(\sin^2 t + \cos^2 t)}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \underline{\underline{\frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}}}$$

Determine parameter t of minimal/maximal curvature:

$$\dot{\kappa}(t) \overset{!}{=} 0 = ab\left(-\frac{3}{\cancel{2}}\right)\frac{a^2\cancel{2}\sin t\cos t - b^2\cancel{2}\cos t\sin t}{(a^2\sin^2 t + b^2\cos^2 t)^{5/2}}$$

$$= \underline{\underline{3ab\frac{(b^2 - a^2)\sin t\cos t}{(a^2\sin^2 t + b^2\cos^2 t)^{5/2}} \overset{!}{=} 0}}$$

$\Rightarrow$ $\underline{\underline{a = b}}$ (circle: curvature is constant!)

$\vee$ $\underline{\underline{\sin t = 0}}$: $\Rightarrow$ $t = n\cdot\pi$, $n \in \mathbb{N}\cup\{0\}$

$\vee$ $\underline{\underline{\cos t = 0}}$ $\Rightarrow$ $t = \frac{(2n+1)\pi}{2}$, $n \in \mathbb{N}\cup\{0\}$

Within $t \in [0, 2\pi)$:

$$\underbrace{\underline{\underline{t_1 = 0}},\ \underline{\underline{t_2 = \frac{\pi}{2}}}},\ \underline{\underline{t_3 = \pi}},\ \underline{\underline{t_4 = \frac{3\pi}{2}}}$$

consider $t_1 = 0$:

$$\kappa_1(t_1 = 0) = \frac{ab}{b^3} = \underline{\underline{\frac{a}{b^2}}}$$

consider $t_2 = \frac{\pi}{2}$:

$$\kappa_2\left(t_2 = \frac{\pi}{2}\right) = \frac{ab}{a^2} = \underline{\underline{\frac{b}{a^2}}}$$

$a > b \Rightarrow \kappa_1 > \kappa_2 \Rightarrow \underline{\underline{t_{\kappa max} = 0}},\ (\pi)$

Aufgabe 3

(i) $r = \phi, \quad \phi \geq 0$:

$$\Rightarrow \quad \vec{r}(t) = t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \text{in 2d polar coordinates}$$

tangent:

$$\dot{\vec{r}}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \Rightarrow \vec{\tau} = t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \lambda \begin{pmatrix} \cos t - t\sin t \\ \sin t + t\cos t \end{pmatrix}$$

(ii) $r = \ln \phi, \quad \phi \geq 1$:

$$\vec{r}(t) = \ln t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad \text{in 2d polar coordinates}$$

tangent:

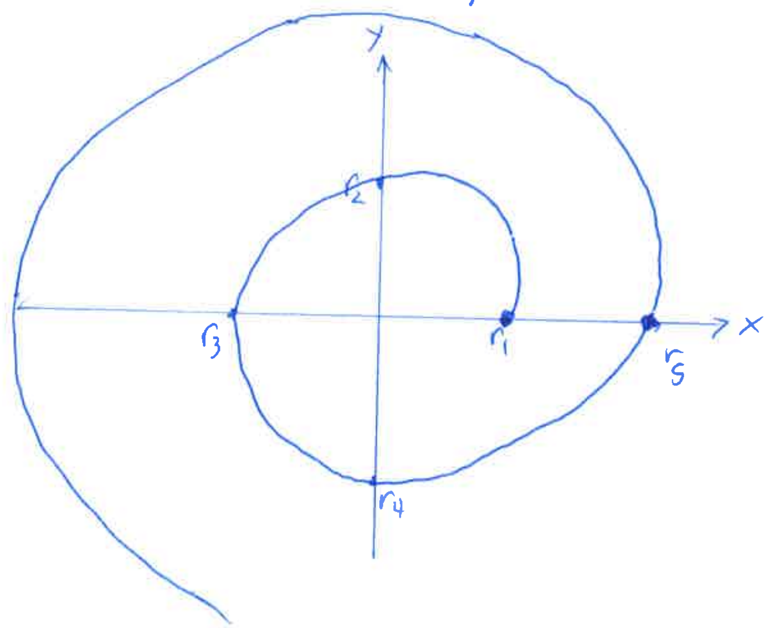
$$\dot{\vec{r}}(t) = \frac{1}{t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \ln t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$\Rightarrow \quad \vec{\tau} = \ln t \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \lambda \left\{ \frac{1}{t} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \ln t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\}$$

Aufgabe 4

$$\vec{r}(t) = e^{at} \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \quad t \geq 0, \quad a > 0$$

(i)



$$\vec{r}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_2\left(\tfrac{\pi}{2}\right) = e^{a\frac{\pi}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{r}_3(\pi) = e^{a\frac{2\pi}{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_4\left(\tfrac{3\pi}{2}\right) = e^{a\frac{3\pi}{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{r}_5(2\pi) = e^{a\frac{4\pi}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

arc length, $t \in [0, 2\pi]$: ($r(t)$ continuous differentiable) (6)

$$S(t) = \int_0^{2\pi} dt \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2}$$

$$r'(t) = a e^{at} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + e^{at} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = e^{at} \begin{pmatrix} a \cos t - \sin t \\ a \sin t + \cos t \end{pmatrix}$$

$$(r'(t))^2 = e^{2at}\left[(a \cos t - \sin t)^2 + (a \sin t + \cos t)^2\right]$$

$$= e^{2at}\left[a^2\cos^2 t - \cancel{2a\cos t \sin t} + \sin^2 t + a^2 \sin^2 t + \cancel{2a \sin t \cos t} + \cos^2 t\right]$$

$$= e^{2at}\left[a^2(\sin^2 t + \cos^2 t) + \sin^2 t + \cos^2 t\right] = \underline{\underline{e^{2at}[a^2+1]}}$$

$$(r(t))^2 = e^{2at}$$

$$\Rightarrow S(t) = \int_0^{2\pi} dt \sqrt{e^{2at}[a^2+1]} = \int_0^{2\pi} dt \sqrt{e^{2at}[1+a^2]}$$

$$= \int_0^{2\pi} dt\, e^{at}\sqrt{a^2+1} = \frac{\sqrt{a^2+1}}{a} e^{at}\Big|_0^{2\pi} = \underline{\underline{\frac{\sqrt{a^2+1}}{a}\left[e^{2\pi a} - 1\right]}}$$

(ii) TNB-frame:

T .... unit vector tangent
N .... normal unit vector
B ... binormal unit vector, $B = T \times N$

$$s(t) = \int_{t_0}^{t} d\tau \, \| \dot{\vec{x}}(\tau) \| \Rightarrow$$

$$\dot{\vec{x}}(t) = \frac{d}{dt}\vec{x}(s(t)) = \vec{x}'(s)\,\dot{s}(t) = \vec{x}'(s)\,\|\dot{\vec{x}}(t)\|$$

$$\Rightarrow \vec{x}'(s) = \frac{\dot{\vec{x}}(t)}{\|\dot{\vec{x}}(t)\|} \Rightarrow \|\vec{x}'(s)\| = 1$$

and $\vec{x}'(s)$ is unit vector tangent $t(s)$

$$\Rightarrow \frac{ds}{dt} = \|\dot{\vec{x}}(t)\| = \|\dot{\vec{r}}(t)\| = \sqrt{e^{2at}[a^2+1]} = \underline{\underline{e^{at}\sqrt{a^2+1}}}$$

$$\Rightarrow t(s) = \frac{d\vec{r}}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{e^{at}}{e^{at}\sqrt{a^2+1}}\begin{pmatrix} a\cos t - \sin t \\ a\sin t + \cos t \\ 0 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{\sqrt{a^2+1}}\begin{pmatrix} a\cos t - \sin t \\ a\sin t + \cos t \\ 0 \end{pmatrix}}}$$

normal unid vector:

$$\vec{r}''(s) = \frac{d}{ds}t(s) = \frac{dt}{ds}\frac{d\,t(s)}{dt} = \overbrace{\frac{1}{e^{at}\sqrt{a^2+1}}}^{\frac{dt}{ds}}\frac{1}{\sqrt{a^2+1}}\frac{d}{dt}\begin{pmatrix} a\cos t - \sin t \\ a\sin t + \cos t \\ 0 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{e^{at}(a^2+1)}\begin{pmatrix} -a\sin t - \cos t \\ a\cos t - \sin t \\ 0 \end{pmatrix}}}$$

$$\|\vec{r}''(s)\| = \frac{e^{-at}}{a^2+1}\left[a^2\sin^2 t + \cos^2 t + \cancel{2a\sin t\cos t} + a^2\cos^2 t + \sin^2 t - \cancel{2a\sin t\cos t}\right]^{\frac{1}{2}}$$

$$= \frac{e^{-at}}{a^2+1}[a^2+1]^{\frac{1}{2}} = \underline{\underline{\frac{1}{e^{at}\sqrt{a^2+1}}}}$$

$$\Rightarrow n(s) = \frac{\vec{r}''(s)}{\|\vec{r}''(s)\|} = \underline{\underline{\frac{1}{\sqrt{a^2+1}}\begin{pmatrix} -a\sin t - \cos t \\ a\cos t - \sin t \\ 0 \end{pmatrix}}}$$

binormal unid vector:

$$b(s) = t(s) \times n(s) = \frac{1}{a^2+1}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ a\cos t - \sin t & a\sin t + \cos t & 0 \\ -a\sin t - \cos t & a\cos t - \sin t & 0 \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ a^2\cos^2 t + \sin^2 t - \cancel{2a\cos t\sin t} + a^2\sin^2 t + \cos^2 t + \cancel{2a\cos t\sin t} \end{pmatrix}\frac{1}{a^2+1} = \frac{1}{a^2+1}\begin{pmatrix} 0 \\ 0 \\ a^2+1 \end{pmatrix} = \%$$

$$\% = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(iii) curvature:

$$\kappa = \|\vec{r}\,''(s)\| = \frac{1}{e^{at}\sqrt{a^2+1}}$$

torsion:

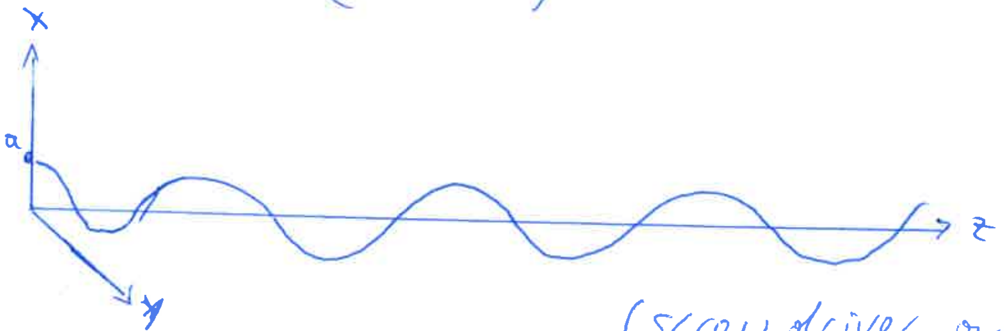
$$\frac{d}{ds} n(s) = \frac{dt}{ds}\frac{dn}{dt} = \overbrace{\frac{1}{e^{at}\sqrt{a^2+1}}}^{\frac{dt}{ds}} \frac{1}{\sqrt{a^2+1}} \begin{pmatrix} -a\cos t + \sin t \\ -a \sin t - \cos t \\ 0 \end{pmatrix}$$

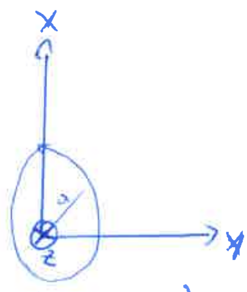
$$\tau(s) = \vec{b} \cdot \frac{d}{ds} n(s) = 0$$

Aufgabe 5

$$\vec{r}(t) = \begin{pmatrix} a\cos(\omega t) \\ a \sin(\omega t) \\ ct \end{pmatrix}, \quad t \geq 0, \quad a, \omega, c > 0$$

(i)





(screw driver around z-axis)

arc length $t \in [0, \frac{2\pi}{\omega}]$: (continuous differentiable)

$$s(t) = \int_0^{2\pi/\omega} dt' \sqrt{\left(\frac{dx}{dt'}\right)^2 + \left(\frac{dy}{dt'}\right)^2 + \left(\frac{dz}{dt'}\right)^2}$$

$$\left.\begin{aligned} \vec{r}^{\,2}(t) &= a^2 + c^2 t^2 \\ \dot{\vec{r}}(t) &= \begin{pmatrix} -a\omega \sin(\omega t) \\ a\omega \cos(\omega t) \\ c \end{pmatrix} \\ \dot{\vec{r}}^{\,2}(t) &= a^2\omega^2 + c^2 \end{aligned}\right\} \Rightarrow \begin{aligned} l &= \int_0^{2\pi/\omega} \sqrt{+a^2\omega^2\sin^2 t + a^2\omega^2\cos^2 t + c^2}\, dt \\ &= \int_0^{2\pi/\omega} \sqrt{a^2\omega^2 + c^2}\, dt \\ &= \sqrt{a^2\omega^2 + c^2}\, \frac{2\pi}{\omega} \end{aligned}$$

(ii) <u>curvature and normal unit vector</u>

$$\frac{ds}{dt} = \|\dot{\vec{r}}(t)\| = \sqrt{a^2\omega^2 + c^2} \quad \underline{\text{tangent}}$$

$$t(s) = \frac{d\vec{r}}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{1}{\sqrt{a^2\omega^2+c^2}}\begin{pmatrix} -a\omega\sin\omega t \\ a\omega\cos\omega t \\ c \end{pmatrix}$$

<u>curvature:</u>

$$\kappa = \|\vec{r}''(s)\|$$

$$\vec{r}''(s) = \frac{d}{ds}t(s) = \frac{dt}{ds}\frac{d\,t(s)}{dt} = \frac{1}{\sqrt{a^2\omega^2+c^2}}\frac{1}{\sqrt{a^2\omega^2+c^2}}\begin{pmatrix} -a\omega^2\cos\omega t \\ -a\omega^2\sin\omega t \\ 0 \end{pmatrix}$$

$$= \frac{-a\omega^2}{a^2\omega^2+c^2}\begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}$$

$$\|\vec{r}''(s)\| = \frac{1}{a^2\omega^2+c^2}\sqrt{a^2\omega^4} = \frac{a\omega^2}{a^2\omega^2+c^2} = \kappa$$

<u>normal unit vector:</u>

$$n(s) = \frac{\vec{r}''(s)}{\|\vec{r}''(s)\|} = -\frac{(a\cancel{\omega^2}+c^2)\,a\omega^2}{(a^2\cancel{\omega^2}+c^2)\,a\omega^2}\begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} = -\begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}$$

(iii) <u>TNB frame and torsion</u>

$$b(s) = t(s) \times n(s) = \frac{1}{\sqrt{a^2\omega^2+c^2}}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ -a\omega\sin\omega t & a\omega\cos\omega t & c \\ -\cos\omega t & -\sin\omega t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2\omega^2+c^2}}\begin{pmatrix} c\sin\omega t \\ -c\cos\omega t \\ a\omega \end{pmatrix}$$

torsion

$$\frac{d}{ds} n(s) = \frac{dt}{ds}\frac{dn}{dt} = \frac{-1}{\sqrt{a^2\omega^2+c^2}}\begin{pmatrix}-\omega\sin\omega t\\ \omega\cos\omega t\\ 0\end{pmatrix}$$

$$\tau(s) = b \cdot \frac{d}{ds} n(s) = \frac{1}{a^2\omega^2+c^2}\left[\omega c\right] = \underline{\underline{\frac{\omega c}{a^2\omega^2+c^2}}}$$