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# Vektoranalysis (für PhysikerInnen)

SS 2014

1. Übungsblatt

11. März

Aufgabe 1
Berechnen Sie folgende unbestimmte Integrale.

(i) $\int \frac{1}{x^3(x+2)} dx$;

(ii) $\int \frac{-75x+113}{(x+3)(x^2+4)^2} dx$.

Aufgabe 2

(i) Berechnen Sie das unbestimmte Integral $\int (1+3x^2)\ln(x-x^2)dx$ für $0 < x < 1$.

(ii) Berechnen Sie das bestimmte Integral $\int_0^{2\pi} e^{2x}\sin^2 x dx$.

Aufgabe 3
Berechnen Sie die folgenden bestimmten Integrale.

(i) $\int_{-\frac{3}{8}}^{0} \frac{1}{\sqrt{2-3x-4x^2}} dx$;

(ii) $\int_0^3 \left(e^{3x} - (e^x)^{\frac{1}{3}}\right) dx$.

Aufgabe 4
Bestimmen Sie die Bogenlängen der Kurven

(i) $K = \left\{(x,y)^T \in \mathbb{R}^2 : y = x^2 - \frac{\ln x}{8}, 1 \leq x \leq \sqrt{e}\right\}$ und

(ii) $r(\phi) = a(\cos\phi + \sin\phi)$, $0 \leq \phi \leq \pi/2$, wobei $a > 0$ fest ist.

Aufgabe 5

(i) Bestimmen Sie (falls existent) den Wert des uneigentlichen Integrals $\int_0^1 \frac{\arcsin x}{\sqrt{1-x}} dx$.

(ii) Berechnen Sie den Cauchyschen Hauptwert des Integrals $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cot x dx$.

# Lösungen zum 1. Übungsblatt aus Vektoranalysis SS14 ①

A. Windisch

## Aufgabe 1

(i) $\displaystyle\int \frac{dx}{x^3(x+2)}$

$$\frac{1}{x^3(x+2)} = \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{b}{x+2} \quad \Big| \cdot x^3(x+2) \quad \left(\text{use partial fraction decomposition}\right)$$

$$1 = a_1(x^2(x+2)) + a_2 x(x+2) + a_3(x+2) + bx^3$$

equating coefficients

$$1 \overset{!}{=} a_1 x^3 + a_1 2x^2 + a_2 x^2 + 2a_2 x + a_3 x + 2a_3 + bx^3$$

$\Rightarrow$ $x^3$: $a_1 + b = 0$ $\Rightarrow b = -\frac{1}{8}$

$x^2$: $2a_1 + a_2 = 0$ $\Rightarrow a_1 = \frac{1}{8}$

$x^1$: $2a_2 + a_3 = 0$ $\Rightarrow a_2 = -\frac{1}{4}$

$x^0$: $2a_3 = 1$ $\Rightarrow a_3 = \frac{1}{2}$

$$\Rightarrow \int \frac{dx}{x^3(x+2)} = \frac{1}{8}\int \frac{1}{x}\,dx - \frac{1}{4}\int \frac{1}{x^2}\,dx + \frac{1}{2}\int \frac{1}{x^3}\,dx - \frac{1}{8}\int \frac{1}{x+2}\,dx$$

$$= \underline{\underline{\frac{1}{8}\log x + \frac{1}{4}\frac{1}{x} - \frac{1}{4}\frac{1}{x^2} - \frac{1}{8}\log(x+2) + c}}$$

(ii) $\displaystyle\int \frac{-75x+113}{(x+3)(x^2+4)^2}\,dx$

complex conjugate roots: use this Ansatz for decomposition

$$\frac{-75x+113}{(x+3)(x^2+4)^2} = \frac{a}{x+3} + \frac{b_1+b_2x}{x^2+4} + \frac{b_3+b_4x}{(x^2+4)^2} \quad \Big| \cdot (x+3)(x^2+4)^2$$

$$-75x+113 = ax^4 + 8ax^2 + 16a + (b_1+b_2x)(x^3+3x^2+4x+12) + b_4x^2 + 3b_4x + b_3x + 3b_3$$

$$= ax^4 + b_2x^4 + b_1x^3 + 3b_2x^3 + 8ax^2 + 3b_1x^2 + 4b_2x^2 + b_4x^2 + 4b_1x + b_3x + 12b_2x$$
$$+ 3b_4x + 16a + 3b_3 + 12b_1$$

equating coefficients:

$x^4$: $a + b_2 = 0$
$x^3$: $b_1 + 3b_2 = 0$
$x^2$: $8a + 3b_1 + 4b_2 + b_4 = 0$
$x^1$: $4b_1 + 12b_2 + b_3 + 3b_4 = -75$
$x^0$: $16a + 12b_1 + 3b_3 = 113$

$$\begin{array}{c|c} & \begin{matrix} a & b_1 & b_2 & b_3 & b_4 \end{matrix} \\ \begin{matrix} a \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 8 & 3 & 4 & 0 & 1 \\ 0 & 4 & 12 & 1 & 3 \\ 16 & 12 & 0 & 3 & 0 \end{pmatrix} \end{array} \begin{pmatrix} a \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -75 \\ 113 \end{pmatrix} \hat{=} \quad M\vec{x} = \vec{b}$$

det M:

$$\begin{vmatrix} 1 & 3 & 0 & 0 \\ 3 & 4 & 0 & 1 \\ 4 & 12 & 1 & 3 \\ 12 & 0 & 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 0 \\ 8 & 3 & 0 & 1 \\ 0 & 4 & 1 & 3 \\ 16 & 12 & 3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 0 & 1 \\ 12 & 1 & 3 \\ 0 & 3 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 & 1 \\ 4 & 1 & 3 \\ 12 & 3 & 0 \end{vmatrix} - \begin{vmatrix} 8 & 0 & 1 \\ 0 & 1 & 3 \\ 16 & 3 & 0 \end{vmatrix}$$

$$= \cancel{36} - \cancel{36} - \cancel{36} - 3(-27 - \cancel{12}) + 72 + 16$$
$$= 81 + 72 + 16 = \underline{\underline{169}}$$

det for a:

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ -75 & 4 & 1 & 3 \\ 113 & 12 & 3 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 1 \\ -75 & 1 & 3 \\ 113 & 3 & 0 \end{vmatrix} = 225 + 113 = \underline{\underline{338}}$$

$$\Rightarrow \quad a = \frac{338}{169} = \underline{\underline{2}}$$

$$\Rightarrow \quad \underline{\underline{b_2 = -2}} \quad \Rightarrow \quad b_1 = -3b_2 = \underline{\underline{6}}$$

$$\Rightarrow \quad 16 + 18 - 8 + b_4 = 0 \quad \Rightarrow \quad \underline{\underline{b_4 = -26}}$$

$$\Rightarrow \quad 32 + 72 + 3b_3 = 113 \quad \Rightarrow \quad 3b_3 = 9 \quad \Rightarrow \quad \underline{\underline{b_3 = 3}}$$

The integral becomes:

$$\int \frac{-75x+113}{(x+3)(x^2+4)^2}\,dx = \underbrace{2\int \frac{dx}{x+3}}_{\mathrm{I}} + \underbrace{6\int \frac{dx}{x^2+4}}_{\mathrm{II}} + \underbrace{\int \frac{-2x}{x^2+4}\,dx}_{\mathrm{III}}$$

$$+ \underbrace{3\int \frac{dx}{(x^2+4)^2}}_{\mathrm{IV}} + \underbrace{\int \frac{-26x}{(x^2+4)^2}\,dx}_{\mathrm{V}}$$

$$\mathrm{I}:\ 2\int \frac{dx}{x+3} = 2\log(x+3) + c$$

$$\mathrm{II}:\ 6\int \frac{dx}{x^2+4} = \left| \begin{array}{l} x = 2\tan u = 2\sin u \cos^{-1} u \\ \frac{d}{du}\tan u = \cos u \cos u^{-1} + \sin u(-\cos u^{-2})(-\sin u) \\ \qquad = 1 + \frac{1}{\cos^2 u} - \frac{\cos^2 u}{\cos^2 u} = \frac{1}{\cos^2 u} \\ dx = 2\frac{1}{\cos^2 u}\,du \end{array} \right|$$

$$= 2\cdot 6 \int \frac{\cos^{-2} u}{4(1+\tan^2 u)}\,du = \frac{12}{4}\int \frac{\cos^{-2} u}{\frac{\cos^2 u + \sin^2 u}{\cos^2 u}}\,du = 3\int du = 3u$$

$$= 3\arctan\frac{x}{2} + c$$

$$\mathrm{III}:\ \int \frac{-2x}{x^2+4}\,dx = \left| \begin{array}{l} u = x^2+4 \\ du = 2x\,dx \end{array} \right| = -\int \frac{\cancel{2x}}{u}\frac{du}{\cancel{2x}} = -\log(x^2+4) + c$$

$$\mathrm{IV}:\ 3\int \frac{dx}{(x^2+4)^2} = \left| \begin{array}{l} x = 2\tan u \\ dx = 2\frac{1}{\cos^2 u}\,du \\ \text{(siehe oben)} \end{array} \right| = \frac{6}{16}\int \frac{\frac{1}{\cos^2 u}}{\frac{1}{\cos^4 u}}\,du = \frac{6}{16}\int \cos^2 u\,du$$

$$= \left| \begin{array}{l} \sin^2 u + \cos^2 u = 1 \\ \cos^2 u - \sin^2 u = \cos 2u \\ \Rightarrow \cos^2 u = \frac{1}{2}(\cos 2u + 1) \end{array} \right| = \frac{3}{16}\int \cos 2u\,du + \frac{3}{16}u = \left| \begin{array}{l} 2u = v \\ 2du = dv \end{array} \right|$$

$$= \frac{3}{16}u + \frac{3}{32}\int \cos v\,dv = \frac{3}{16}u + \frac{3}{32}\sin v = \frac{3}{16}\arctan\frac{x}{2} + \frac{3}{32}\sin 2u$$

$$= \frac{3}{16}\arctan\frac{x}{2} + \frac{3}{32}\sin\left(2\cdot\arctan\frac{x}{2}\right) + c$$

$$\mathrm{V}:\ \int \frac{-26x}{(x^2+4)^2}\,dx = \left| \begin{array}{l} u = x^2+4 \\ du = 2x\,dx \\ dx = \frac{du}{2x} \end{array} \right| = -\int \frac{13}{u^2}\,du = 13\frac{1}{u} = 13\frac{1}{x^2+4} + c$$

Solution:



$$\int \frac{-75x+113}{(x+3)(x^2+4)^2}\,dx = \underline{\underline{I + II + III + IV + V + c}}$$

## Aufgabe 2

(i) $\displaystyle\int (1+3x^2)\ln(x-x^2)\,dx \quad , \quad 0<x<1$

$$= \int \ln(x(1-x))\,dx + \int 3x^2 \ln(x(1-x))\,dx$$

$$= \underbrace{\int \ln x\,dx}_{\tilde{I}} + \underbrace{\int \ln(1-x)\,dx}_{\tilde{II}} + \underbrace{\int 3x^2 \ln x\,dx}_{\tilde{III}} + \underbrace{\int 3x^2\ln(1-x)\,dx}_{\tilde{IV}}$$

$\tilde{I}$: $\displaystyle\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\cdot\ln x - \int x\cdot\frac{1}{x}\,dx = \underline{\underline{x\cdot\ln x - x + c}}$

$\tilde{II}$: $\displaystyle\int \ln(1-x)\,dx = -\int \ln u\,du = -u\ln u + u = \underline{\underline{-(1-x)\ln(1-x) + (1-x) + c}}$

$\tilde{III}$: $\displaystyle\int 3x^2 \ln x\,dx = (x\cdot\ln x - x)\,3x^2 - 6\int (x\cdot\ln x - x)\,x\,dx$

$$= 3x^3\ln x - 3x^3 + 2x^3 - 6\int x^2 \ln x\,dx$$

$$\Rightarrow \int 9x^2 \ln x\,dx = 3x^3\ln x - x^3 + c$$

$$\Rightarrow \int 3x^2\ln x\,dx = \underline{\underline{x^3\ln x - \frac{x^3}{3} + c}}$$

$\tilde{IV}$: $\displaystyle\int 3x^2\ln(1-x)\,dx = \underbrace{[-(1-x)\ln(1-x) + (1-x)]\,3x^2}_{=A} - 6\int [-(1-x)\ln(1-x) + (1-x)]\,x\,dx$

$$= A + 6\int x\ln(1-x)\,dx - 6\int x^2\ln(1-x)\,dx - 6\int x(1-x)\,dx$$

$$= A + 6\int x\ln(1-x)\,dx - 6\int x^2\ln(1-x)\,dx \underbrace{- 3x^2 + 2x^3}_{=B}$$

$$\Rightarrow 9\int x^2 \ln(1-x)\,dx = A + B + 6\int x\ln(1-x)\,dx \quad (*)$$

$$\Rightarrow 6\int x\ln(1-x)\,dx = 6x\left(-(1-x)\ln(1-x)+(1-x)\right) - 6\int\left(-(1-x)\ln(1-x)+(1-x)\right)dx$$

$$= -6\ln(1-x) + 6x\ln(1-x) + 6 - 6x + 6\int \ln(1-x)\,dx - 6\int x\ln(1-x)\,dx$$

$$-6\int dx + 6\int x\,dx$$

$$\Rightarrow 12\int x\ln(1-x)\,dx = -6x\ln(1-x) + 6x^2\ln(1-x) + \cancel{6x} - \cancel{6}^{3}x^2$$

$$+6\left(-(1-x)\ln(1-x)+(1-x)\right) \cancel{-6x} + \cancel{3x^2}$$

$$= -3x^2 + 6x^2\ln(1-x) - \cancel{6x\ln(1-x)} - 6\ln(1-x)$$

$$+ 6x\cancel{\ln(1-x)} + 6 - 6x$$

$$\Rightarrow 6\int x\ln(1-x)\,dx = \underline{\underline{-\frac{3}{2}x^2 - 3x + 3x^2\ln(1-x) - 3\ln(1-x) + c}}$$

$\Rightarrow$ Resubstitude in $(*)$ $\Rightarrow$

$$9\int x^2\ln(1-x)\,dx = -x^3 + 3x^3\ln(1-x) - 3x^2\ln(1-x) + 6\int x\ln(1-x)\,dx$$

$$= -x^3 + 3x^3\ln(1-x) - 3x^2\ln(1-x) - \frac{3}{2}x^2 + 3x^2\ln(1-x)$$

$$-3\ln(1-x) - 3x$$

$$= -x^3 - \frac{3}{2}x^2 - 3x + 3x^3\ln(1-x) - 3\ln(1-x) + c$$

$$\Rightarrow \widetilde{IV}:\ 3\int x^2\ln(1-x)\,dx = \underline{\underline{-\frac{x^3}{3} - \frac{x^2}{2} - x + x^3\ln(1-x) - \ln(1-x) + c}}$$

Solution:

$$\int (1+3x^2)\ln(x-x^2)\,dx \qquad (0<x<1)$$

$$= \underline{\underline{\widetilde{I} + \widehat{II} + \widehat{III} + \widetilde{IV} + c}}$$

(ii) $\int_0^{2\pi} e^{2x}\sin^2 x\,dx = \int_0^{2\pi} e^{2x}(1-\cos^2 x)\,dx = \int_0^{2\pi} e^{2x}dx - \int_0^{2\pi} e^{2x}\cos^2 x\,dx$ (6)

$$= \frac{1}{2}e^{2x}\Big|_0^{2\pi} - \int_0^{2\pi} e^{2x}\frac{1}{2}(\cos(2x)+1)\,dx$$

$$= \frac{1}{2}e^{2x}\Big|_0^{2\pi} - \frac{1}{4}e^{2x}\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^{2x}\cos(2x)\,dx$$

$$= \underbrace{\frac{1}{2}e^{4\pi} - \frac{1}{2} - \frac{1}{4}e^{4\pi} + \frac{1}{4}}_{=A} - \frac{1}{2}\int_0^{2\pi} e^{2x}\cos(2x)\,dx$$

$$\int_0^{2\pi} e^{2x}\cos(2x)\,dx = \frac{1}{2}\cos(2x)e^{2x}\Big|_0^{2\pi} + \int_0^{2\pi}\sin(2x)e^{2x}\,dx$$

$$= \frac{1}{2}\cos(2x)e^{2x}\Big|_0^{2\pi} + \underbrace{\cancel{\frac{1}{2}\sin(2x)e^{2x}\Big|_0^{2\pi}}}_{=0} - \int_0^{2\pi} e^{2x}\cos(2x)\,dx$$

$$\Rightarrow 2\int_0^{2\pi} e^{2x}\cos(2x)\,dx = \frac{1}{2}e^{4\pi} - \frac{1}{2}$$

$$\Rightarrow \int_0^{2\pi} e^{2x}\sin^2 x\,dx = A - \frac{1}{8}e^{4\pi} + \frac{1}{8}$$

$$= \frac{1}{2}e^{4\pi} - \frac{1}{2} - \frac{1}{4}e^{4\pi} + \frac{1}{4} - \frac{1}{8}e^{4\pi} + \frac{1}{8}$$

$$= \underline{\underline{\frac{1}{8}e^{4\pi} - \frac{1}{8}}}$$

# Aufgabe 3

(i)
$$\int_{-\frac{3}{8}}^{0} \frac{1}{\sqrt{2-3x-4x^2}}\,dx = \int_{-\frac{3}{8}}^{0} \frac{dx}{\sqrt{-(2x+\frac{3}{4})^2+\frac{9}{16}+2}} = \int_{-\frac{3}{8}}^{0} \frac{dx}{\sqrt{-(2x+\frac{3}{4})^2+\frac{41}{16}}}$$

$$= \int_{-\frac{3}{8}}^{0} \frac{dx}{\sqrt{\frac{41}{16}}\sqrt{1-\frac{16}{41}(2x+\frac{3}{4})^2}} = \frac{4}{\sqrt{41}}\int_{-\frac{3}{8}}^{0} \frac{dx}{\sqrt{1-\left(\frac{8x}{\sqrt{41}}+\frac{12}{4\sqrt{41}}\right)^2}}$$

$$= \left|\begin{array}{l} u = \frac{8}{\sqrt{41}}x + \frac{12}{4\sqrt{41}} \\ du = \frac{8}{\sqrt{41}}dx \\ -\frac{3}{8} \to -\frac{3}{\sqrt{41}} + \frac{3}{\sqrt{41}} = 0 \\ 0 \to \frac{3}{\sqrt{41}} \end{array}\right| = \underbrace{\frac{4}{\sqrt{41}}\frac{\sqrt{41}}{8}}_{\frac{1}{2}}\int_{0}^{\frac{3}{\sqrt{41}}} \frac{du}{\sqrt{1-u^2}} = \left|\begin{array}{l} u = \sin\vartheta \\ du = \cos\vartheta\, d\vartheta \end{array}\right|$$

$$= \frac{1}{2}\int \frac{\cos\vartheta\, d\vartheta}{\cos\vartheta} = \frac{1}{2}\arcsin u \Big|_{0}^{\frac{3}{\sqrt{41}}} = \underline{\underline{\frac{1}{2}\arcsin\frac{3}{\sqrt{41}}}}$$

(ii)
$$\int_{0}^{3}\left(e^{3x}-(e^x)^{\frac{1}{3}}\right)dx = \int_{0}^{3} e^{3x}dx - \int_{0}^{3} e^{\frac{x}{3}}dx = \left|\begin{array}{l} 3x = u \\ du = 3dx \\ 0 \to 0 \\ 3 \to 9 \\ \frac{x}{3} = v \\ dv = \frac{1}{3}dx \\ 0 \to 0 \\ 3 \to 1 \end{array}\right|$$

$$= \frac{1}{3}\int_{0}^{9} e^u\,du - 3\int_{0}^{1} e^v\,dv = \frac{1}{3}e^9 - \frac{1}{3} - 3e + 3$$

$$= \underline{\underline{\frac{1}{3}\left(e^9 - 9e + 8\right)}}$$

# Aufgabe 4

(i) $f(x) = x^2 - \frac{1}{8}\ln x$ ; $f(x)$ continuously differentiable on $[1, \sqrt{e}] \Rightarrow$

$f(x)$ rectifiable $\Rightarrow$ arc length: $l = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

$\Rightarrow f'(x) = 2x - \frac{1}{8x}$

$$l = \int_1^{\sqrt{e}} \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2}\,dx$$

$$1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$= 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

$$= \left(2x + \frac{1}{8x}\right)^2$$

$$\Rightarrow \quad l = \int_1^{\sqrt{e}} \sqrt{\left(2x + \frac{1}{8x}\right)^2}\,dx$$

$$= \int_1^{\sqrt{e}} \left(2x + \frac{1}{8x}\right) dx$$

$$= x^2 \Big|_1^{\sqrt{e}} + \frac{1}{8} \ln x \Big|_1^{\sqrt{e}}$$

$$= e - 1 + \frac{1}{16}$$

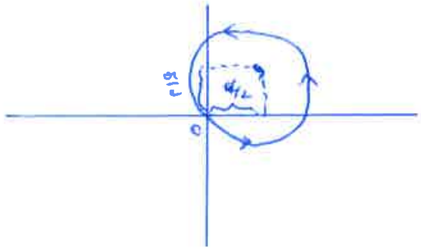
$$= \underline{\underline{e - \frac{15}{16}}}$$

(ii) $r(\phi) = a(\cos\phi + \sin\phi), \quad 0 \le \phi \le \frac{\pi}{2} \qquad a > 0$, ~~fest~~

$$\ell = \int_0^{\frac{\pi}{2}} \sqrt{a^2(\cos\phi + \sin\phi)^2 + a^2(-\sin\phi + \cos\phi)^2}\, d\phi$$

$$= a \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\phi + \sin^2\phi + \cancel{2\sin\phi\cos\phi} + \sin^2\phi + \cos^2\phi - \cancel{2\sin\phi\cos\phi}}\, d\phi$$

$$= a \int_0^{\frac{\pi}{2}} \sqrt{2}\, d\phi = \underline{\underline{a\sqrt{2}\,\frac{\pi}{2}}}$$



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## Aufgabe 5

(i) $\displaystyle\int_0^1 \frac{\arcsin x}{\sqrt{1-x}}\, dx$

Determine antiderivative:

$$\tilde{I} = \int \frac{\arcsin x}{\sqrt{1-x}}\, dx = 2\sqrt{1-x}\,\arcsin x + 2\int \frac{\sqrt{1-x}}{\sqrt{1-x^2}}\, dx$$

$$= -2\sqrt{1-x}\,\arcsin x + 2\int \frac{1}{\sqrt{1+x}}\, dx = \underline{\underline{-2\sqrt{1-x}\,\arcsin x + 4\sqrt{1+x} + c}}$$

Determine improper integral:

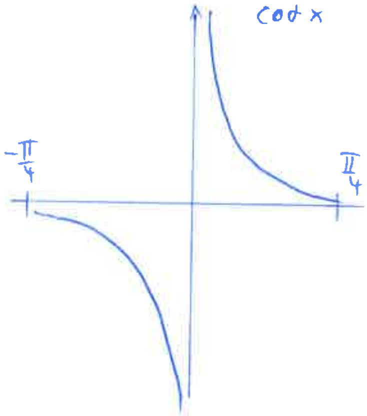
$$I = \lim_{B \to 1} \int_0^B \frac{\arcsin x}{\sqrt{1-x}}\, dx$$

$$= \lim_{B \to 1} \left(-2\sqrt{1-B}\,\arcsin B + 4\sqrt{1+B}\right) - 4$$

$$= 4\sqrt{2} - 4$$

$$= \underline{\underline{4(\sqrt{2} - 1)}}$$

(ii) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cot x \, dx$




antiderivative $\ln|\sin(x)|$

integral improper at 0.

$$CH \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cot x \, dx = \lim_{\varepsilon \to 0^+} \left( \int_{-\frac{\pi}{4}}^{0-\varepsilon} \cot x \, dx + \int_{0+\varepsilon}^{\frac{\pi}{4}} \cot x \, dx \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( \ln(|\sin(x)|) \Big|_{-\frac{\pi}{4}}^{-\varepsilon} + \ln(|\sin(x)|) \Big|_{\varepsilon}^{\frac{\pi}{4}} \right)$$

$$= \lim_{\varepsilon \to 0^+} \left( \cancel{\ln(\sin(\varepsilon))} - \cancel{\ln(\sin(\tfrac{\pi}{4}))} + \cancel{\ln(\sin(\tfrac{\pi}{4}))} - \cancel{\ln(\sin(\varepsilon))} \right)$$

$$= \lim_{\varepsilon \to 0^+} (0) = \underline{\underline{0}}$$