BPHZ-ing the Squint Diagram in Landau gauge QCD

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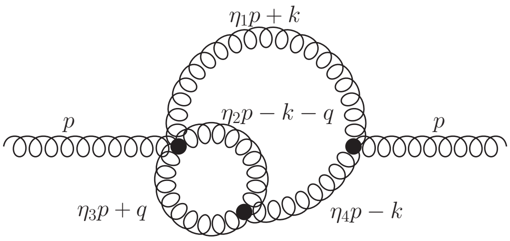
The BPHZ scheme is applied to the Squint diagram in order to render the amplitude finite. This is done in a perturbative regime, the actual calculation is not performed; the result is derived symbolically. Nevertheless, a full prescription of how to obtain the finite amplitude that corresponds to the Squint diagram is provided.

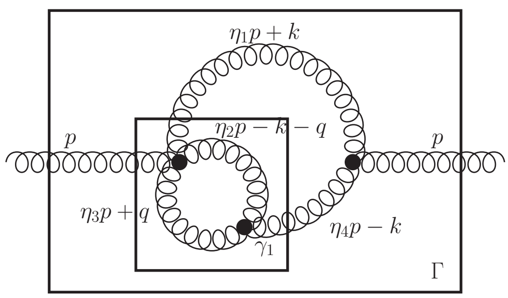
I. BPHZ-ING THE SQUINT

Let us apply the BPHZ procedure [1–4] to the so-called Squint-Diagram in Ladau gauge QCD. The BPHZ procedure allows one to render Feynman amplitudes of in principle arbitrary kind and degree of divergence finite. To learn about BPHZ, see i.e. the references given above, or consult one of the textbooks [5–9] or the article of [10] or [11]. For readers interested in the underlying mathematical structure of renormalization in perturbative Quantum Field Theory, the textbook of [12] might be found especially delightful. The procedure acts on level of the integrands of the Feynman diagrams. Without going into the details of the procedure, let us just apply it to the Squint diagram now. The diagram is given by

$\Gamma =$ 

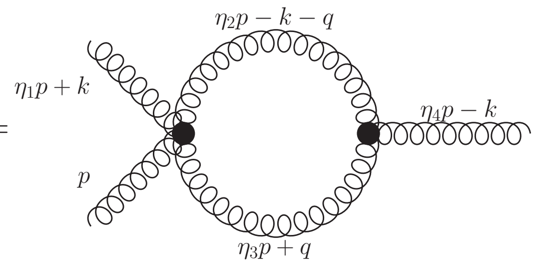
(1)

The momentum flows from the left to the right, as depicted in (1). When using BPHZ and Zimmermann's so-called *Forest Formula*, the first step is to determine the *Family of Forests*. The family of forests is a set containg the empty set, the whole diagram, as well as all of its forests. A forest is a renormalization part of the diagram, i.e. any proper (1PI), superficially divergent subdiagram of the whole diagram, as well as any possible unifications of (sub)diagrams which do not overlap. In the case of the Squint, we have the following forests, which are depicted in (2) by rectangular boxes.

$\Gamma =$ 

(2)

There is only one *subdivergence*, which is *nested* into the *overall divergence* of the whole diagram. The subdivergence of the diagram Γ, let us call it γ_1, is shown in (3).

$\gamma_1 =$ 

(3)

Now we have everything we need in order to apply the mechanism. The family of forests is given by

$$\mathscr{F} = \{\emptyset, \{\Gamma\}, \{\gamma_1\}, \{\Gamma, \gamma_1\}\}, \tag{4}$$

where the empty set and the whole diagram is always part of the family of forests due to the chosen convention. The finite amplitude is given by the forest formula,

$$I_\Gamma^{reg} = \sum_{U \in \mathscr{F}} \prod_{\xi \in U} (-t_\xi^{d(\xi)}) I_\Gamma. \tag{5}$$

Hereby, I_Γ^{reg} is the regularized integrand and I_Γ the unregularized integrand. The $t_\xi^{d(xi)}$ operation is the Taylor expansion in an *external parameter* (usually p, but one can also use the mass m or other external parameters) about an in principle arbitrary expansion point (usually $p = 0$) of the diagram ξ up to the order of its superficial degree of divergence $d(\xi)$, which can be obtained by power counting. Thus, the operation $t_\xi^{d(\xi)}$ is

$$t_\xi^{d(\xi)} = \sum_{i=0}^{d(\xi)} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial p^i} \right]_{p=0}. \tag{6}$$

The meaning of the formulas above will become clear below. First of all observe that the Squint diagram possesses a 4-vertex and two 3-vertices. Note that each 3-vertex contributes one order to the superficial degree of divergence. Furthermore, there are four propagators involved and two loop momenta are to be integrated. Thus, the powercounting of the whole diagram, Γ, yields

$$\frac{\mathscr{O}(4+4+1+1)}{\mathscr{O}(2+2+2+2)} = \mathscr{O}(2), \tag{7}$$

Building Block	**Symbol**
4-vertex	V_4
3-vertex	$V_3(p)$
propagator	$TP(p)\cdot\sigma(p)$

Table I. The building blocks of the integrands: we don't use Feynman rules but the symbols shown in the table to derive the finite amlitude. Hereby TP represents the transverse projector of the gluon propagator, and σ represents the scalar part. Observe that while the 4-vertex does not possess any momentum dependence, this is not the case for the 3-vertex. Thus, the 3-vertex has an effect on the superficial degree of divergence of each diagram it is part of (see text).

i.e. it diverges quadratically, or in other words, its superficial degree of divergence is $d(\Gamma) = 2$. Hereby, the numerator has contributions from 4 times the k integral measure, 4 times the q integral measure, and a contribution of order 1 of each of the 3-vertices. The denominator possesses contributions of the 4 propagators, each giving an order of 2. Thus, in total we are left with a quadratic divergence. In a similar manner we proceed with the subdiagram γ_1, as shown in (3):

$$\frac{\mathscr{O}(4+1)}{\mathscr{O}(2+2)} = \mathscr{O}(1), \tag{8}$$

so it obviously diverges linearly. Suppose we obtained the integrand I_Γ from Feynman rules. Then the forest formula tells us exactly how to proceed:

$$I_\Gamma^{reg} = (1 + (-t_\Gamma^{d(\Gamma)}) + (-t_{\gamma_1}^{d(\gamma_1)}) + (-t_\Gamma^{d(\Gamma)})(-t_{\gamma_1}^{d(\gamma_1)}))I_\Gamma. \tag{9}$$

The one stems from the action of t on the empty set, which yields 1 by definition. We have to clarify how exactly $t_{\gamma_1}^{d(\gamma_1)}$ acts on I_Γ. Following the definition of [5], we have

$$(-t_{\gamma_1}^{d(\gamma_1)})I_\Gamma = I_{\Gamma/\gamma_1}(-t^{d(\gamma_1)})I_{\gamma_1}, \tag{10}$$

where I_{Γ/γ_1} is the integrand obtained from applying Feynman rules to the diagram one gets if the subdiagram γ_1 of Γ is shrunken to a point. Making use of (10), we get

$$\begin{aligned} I_\Gamma^{reg} &= (1 + (-t_\Gamma^{d(\Gamma)}) + (-t_{\gamma_1}^{d(\gamma_1)}) + (-t_\Gamma^{d(\Gamma)})(-t_{\gamma_1}^{d(\gamma_1)}))I_\Gamma \\ &= I_\Gamma - t^2 I_\Gamma - I_{\Gamma/\gamma_1}(t^1 I_{\gamma_1}) + (t^2(I_{\Gamma/\gamma_1}(t^1 I_{\gamma_1}))). \end{aligned} \tag{11}$$

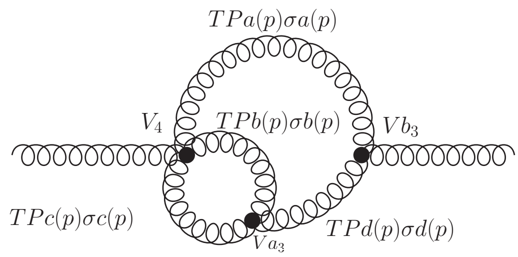
In order to perform the last equation, we will have to express the integrands of the diagrams involved at least symbolically, i.e. without applying actual Feynman rules. The symbols needed for the derivation are summarized in Table I. They allow us to write down the following relevant expressions:

$$\begin{aligned} I_\Gamma = {}& V_4 \cdot Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPb(p)\sigma b(p) \\ & \times TPc(p)\sigma c(p) \cdot TPd(p)\sigma d(p) \end{aligned} \tag{12}$$

$$I_{\gamma_1} = V_4 \cdot Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \tag{13}$$

$$I_{\Gamma/\gamma_1} = V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \tag{14}$$

The meaning of the a's, b's and c's is depicted in (15). The simply label the propagators/vertcies.

$\Gamma =$ 

(15)

The external legs are drawn, but they are not relevant for the calculation. Now let us plug equations (12-14) into the second line of equation (11). This gives us the prescription of how to obtain the finite integrand.

$$\begin{aligned} I_\Gamma^{reg} &= I_\Gamma - t^2 I_\Gamma - I_{\Gamma/\gamma_1}(t^1 I_{\gamma_1}) + (t^2(I_{\Gamma/\gamma_1}(t^1 I_{\gamma_1}))) \qquad (16) \\ &= V_4 \cdot Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPb(p)\sigma b(p) \\ &\quad \times TPc(p)\sigma c(p) \cdot TPd(p)\sigma d(p) \\ &- \sum_{i=0}^{2} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial p^i}\right]_{p=0} V_4 \cdot Va_3(p) \cdot Vb_3(p) \\ &\quad \times TPa(p)\sigma a(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\ &\quad \times TPd(p)\sigma d(p) \\ &- V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\ &\quad \times \sum_{i=0}^{1} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial p^i}\right]_{p=0} \\ &\quad \times V_4 \cdot Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\ &+ \left(\sum_{i=0}^{2} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial p^i}\right]_{p=0} \left(\right.\right. \\ &\quad \times V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\ &\quad \times \left(\sum_{i=0}^{1} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial_p^i}\right]_{p=0} \right. \\ &\quad \left.\left.\left. \times V_4 \cdot Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \right)\right)\right) \end{aligned}$$

In equation (16) we now have to perform the derivatives which are present due to the Taylor expansion. Let us do this in a step by step manner.

First, let us write out the sums. The zeroth order corresponds to rewriting the term it acts on with all external momenta p put to zero. Furthermore, the 4-vertex shows up in every term. Since it is momentum independent, we can drag it to the front. However, unfortunately this is not true for the 3-vertex or the propagators.

There is nothing to be done for the first term, but let us rewrite the whole story for sake of completeness. The whole expression then becomes:

$$
\begin{aligned}
I_\Gamma^{reg} &= V_4 \Bigg[Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \\
&\times\ TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \cdot TPd(p)\sigma d(p) \qquad (17) \\
&-\ Va_3(0) \cdot Vb_3(0) \cdot TPa(0)\sigma a(0) \cdot TPb(0)\sigma b(0) \\
&\times\ TPc(0)\sigma c(0) \cdot TPd(0)\sigma d(0) \\
&-\ p \left[\frac{\partial}{\partial p} \Big(Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p). \right. \\
&\times\ TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\
&\times\ \left. TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&-\ \frac{p^2}{2} \left[\frac{\partial^2}{\partial p^2} \Big(Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p). \right. \\
&\times\ TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\
&\times\ \left. TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&-\ V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times\ Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&-\ V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times\ p \left[\frac{\partial}{\partial p} \Big(Va_3(p) \cdot TPb(p)\sigma b(p) \right. \\
&\times\ \left. TPc(p)\sigma c(p) \Big) \right]_{p=0} \\
&+\ \sum_{i=0}^{2} \frac{p^i}{i!} \left[\frac{\partial^i}{\partial p^i} \right]_{p=0} \\
&\Big(V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times\ Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&+\ V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times\ p \left[\frac{\partial}{\partial p} \left(Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \right) \right]_{p=0} \\
&\Big) \Bigg].
\end{aligned}
$$

I do not resolve the derivatives of equation (17) to any deeper level at this point, but let us perform the second order Taylor expansion which is still present, acting on the last terms of the whole expression. Resolving the sum, we get the (18).

$$
\begin{aligned}
I_\Gamma^{reg} &= V_4 \Bigg[Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \qquad (18) \\
&\times TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \cdot TPd(p)\sigma d(p) \\
&-\ Va_3(0) \cdot Vb_3(0) \cdot TPa(0)\sigma a(0) \cdot TPb(0)\sigma b(0) \\
&\times TPc(0)\sigma c(0) \cdot TPd(0)\sigma d(0) \\
&-\ p \left[\frac{\partial}{\partial p} \Big(Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p). \right. \\
&\times TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\
&\times \left. TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&-\ \frac{p^2}{2} \left[\frac{\partial^2}{\partial p^2} \Big(Va_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p). \right. \\
&\times TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \\
&\times \left. TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&-\ V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&-\ V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \cdot TPd(p)\sigma d(p) \\
&\times p \left[\frac{\partial}{\partial p} \Big(Va_3(p) \cdot TPb(p)\sigma b(p) \right. \\
&\times \left. TPc(p)\sigma c(p) \Big) \right]_{p=0} \\
&+\ V\tilde{a}_3(0) \cdot Vb_3(0) \cdot TPa(0)\sigma a(0) \cdot TPd(0)\sigma d(0) \\
&\times Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&+\ Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&\times p \left[\frac{\partial}{\partial p} \Big(V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \right. \\
&\times \left. \cdot TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&+\ \left[\frac{\partial}{\partial p} \left(Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \right) \right]_{p=0} \\
&\times p \left[\frac{\partial}{\partial p} \Big(p \cdot V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \right. \\
&\times \left. \cdot TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&+\ Va_3(0) \cdot TPb(0)\sigma b(0) \cdot TPc(0)\sigma c(0) \\
&\times \frac{p^2}{2} \left[\frac{\partial^2}{\partial p^2} \Big(V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \right. \\
&\times \left. \cdot TPd(p)\sigma d(p) \Big) \right]_{p=0} \\
&+\ \left[\frac{\partial}{\partial p} \left(Va_3(p) \cdot TPb(p)\sigma b(p) \cdot TPc(p)\sigma c(p) \right) \right]_{p=0} \\
&\times \frac{p^2}{2} \left[\frac{\partial^2}{\partial p^2} \Big(p \cdot V\tilde{a}_3(p) \cdot Vb_3(p) \cdot TPa(p)\sigma a(p) \right. \\
&\times \left. \cdot TPd(p)\sigma d(p) \Big) \right]_{p=0} \Bigg].
\end{aligned}
$$

If one wants to actually calculate the derivatives now, this becomes a really painful job. On this symbolic level this is not a problem, but painful due to the fact that the result is quite lengthy.

We have found the finite amplitude which is now given by equation (18). Of course one still has to replace the rather symbolic expressions by the ones obtained from Feynman rules and perform the derivatives on them. However, at a symbolic level we have found the finite expression corresponding to the finite amplitude of the Squint diagram. The BPHZ scheme ensures cutoff-independence of the result.

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